

Hyperbolic localization and Lefschetz fixed point formulas for higher-dimensional fixed point sets*

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Abstract

We study Lefschetz fixed point formulas for constructible sheaves with higher-dimensional fixed point sets. Under fairly weak assumptions, we prove that the local contributions from them are expressed by some constructible functions associated to hyperbolic localizations. This gives an affirmative answer to a conjecture of Goresky-MacPherson [8] in particular for smooth fixed point components (see [9, page 9, (1.12) Open problems]). In the course of the proof, the new Lagrangian cycles introduced in our previous paper [21] will be effectively used. Moreover we show various examples for which local contributions can be explicitly determined by our method.

1 Introduction

Lefschetz fixed point formulas are important in many branches of mathematics such as topology, algebraic geometry, number theory, dynamical systems and representation theory. Despite a lot of activities on this subject, the case where the fixed point set is higher-dimensional still remains quite mysterious. In this paper we study Lefschetz fixed point formulas for morphisms $\phi: X \rightarrow X$ of real analytic manifolds X whose fixed point set $M = \{x \in X \mid \phi(x) = x\} \subset X$ is higher-dimensional (since we mainly consider the case where the fixed point set is a smooth submanifold of X , we use the symbol M to express it). It is well-known that when X is compact the global Lefschetz number of ϕ

$$\mathrm{tr}(\phi) := \sum_{j \in \mathbb{Z}} (-1)^j \mathrm{tr}\{H^j(X; \mathbb{C}_X) \xrightarrow{\phi^*} H^j(X; \mathbb{C}_X)\} \in \mathbb{C} \quad (1.1)$$

is expressed as the integral of a local cohomology class $C(\phi) \in H_M^n(X; or_X)$ supported by M , where we set $\dim X = n$ and or_X is the orientation sheaf of X (see Dold [4], [5] etc.).

***2010 Mathematics Subject Classification:** 14C17, 14C40, 32C38, 35A27, 37C25, 55N33

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Let $M = \bigsqcup_{i \in I} M_i$ be the decomposition of M into connected components and

$$H_M^n(X; or_X) = \bigoplus_{i \in I} H_{M_i}^n(X; or_X), \quad (1.2)$$

$$C(\phi) = \bigoplus_{i \in I} C(\phi)_{M_i} \quad (1.3)$$

the associated direct sum decompositions. We call the integral $c(\phi)_{M_i} \in \mathbb{C}$ of the local cohomology class $C(\phi)_{M_i} \in H_{M_i}^n(X; or_X)$ for a fixed point component M_i the local contribution from M_i . Then the global Lefschetz number of ϕ is equal to the sum of the local contributions from M_i 's:

$$\mathrm{tr}(\phi) = \sum_{i \in I} c(\phi)_{M_i}. \quad (1.4)$$

But if the fixed point component M_i is higher-dimensional, it is in general very difficult to compute the local contribution by the following dimensional reason. Let M_i be a fixed point component of ϕ whose codimension d in X satisfies the condition $0 < d < n$. Then the local cohomology group $H_{M_i}^n(X; or_X)$ is isomorphic to the 0-dimensional Borel-Moore homology group $H_0^{\mathrm{BM}}(M_i; \mathbb{C})$ of M_i by the Alexander duality, and the class $C(\phi)_{M_i}$ in it cannot be calculated locally at each point of M_i . On the other hand, top-dimensional Borel-Moore homology cycles in M_i , i.e. elements in $H_{n-d}^{\mathrm{BM}}(M_i; \mathbb{C})$ can be much more easily handled since they are realized as sections of a relative orientation sheaf on M_i .

In this paper, we overcome this difficulty partially by using our new Lagrangian cycles introduced in [21]. Since we also want to study Lefschetz fixed point formulas over singular varieties (and those for intersection cohomology groups), from now we consider the following very general setting. Let X , ϕ and $M = \bigsqcup_{i \in I} M_i$ be as before, and F a bounded complex of sheaves of \mathbb{C}_X -modules whose cohomology sheaves are \mathbb{R} -constructible in the sense of [17]. Assume that we are given a morphism $\Phi: \phi^{-1}F \rightarrow F$ in the derived category $\mathbf{D}_{\mathbb{R}-c}^b(X)$. If the support $\mathrm{supp}(F)$ of F is compact, we can define the global trace (Lefschetz number) $\mathrm{tr}(F, \Phi) \in \mathbb{C}$ of the pair (F, Φ) by

$$\mathrm{tr}(F, \Phi) := \sum_{j \in \mathbb{Z}} (-1)^j \mathrm{tr}\{H^j(X; F) \xrightarrow{\Phi} H^j(X; F)\} \in \mathbb{C}, \quad (1.5)$$

where the morphisms $H^j(X; F) \xrightarrow{\Phi} H^j(X; F)$ are induced by

$$F \rightarrow R\phi_* \phi^{-1}F \xrightarrow{\Phi} R\phi_* F. \quad (1.6)$$

In this very general setting, Kashiwara [16] introduced local contributions $c(F, \Phi)_{M_i} \in \mathbb{C}$ from the fixed point components M_i and proved the equality

$$\mathrm{tr}(F, \Phi) = \sum_{i \in I} c(F, \Phi)_{M_i}. \quad (1.7)$$

Therefore the remaining task for us is to calculate the local contributions $c(F, \Phi)_{M_i} \in \mathbb{C}$ explicitly. Let M_i be a fixed point component of ϕ whose regular part $(M_i)_{\mathrm{reg}} \subset M_i$ satisfies the condition $\mathrm{supp}(F) \cap M_i \subset (M_i)_{\mathrm{reg}}$. For the sake of simplicity, we denote $(M_i)_{\mathrm{reg}}$ simply by M . Then there exists a natural morphism

$$\phi': T_M X \rightarrow T_M X \quad (1.8)$$

induced by ϕ , where $T_M X$ is the normal bundle of M in X . For each point $x \in M$, we define a finite subset $\text{Ev}(\phi'_x)$ of \mathbb{C} by

$$\text{Ev}(\phi'_x) := \{\text{the eigenvalues of } \phi'_x: (T_M X)_x \longrightarrow (T_M X)_x\} \subset \mathbb{C}. \quad (1.9)$$

Assume the condition:

$$1 \notin \text{Ev}(\phi'_x) \quad \text{for any } x \in \text{supp}(F) \cap M, \quad (1.10)$$

which means that the graph $\Gamma_\phi = \{(\phi(x), x) \mid x \in X\} \subset X \times X$ of ϕ intersects cleanly (see [17, Definition 4.1.5]) the diagonal set $\Delta_X \subset X \times X$ along $M \subset \Gamma_\phi \cap \Delta_X$ on $\text{supp}(F) \cap M$. It naturally appears also in the classical study of Atiyah-Bott type Lefschetz theorems by Gilkey [7, Theorem 3.9.2], Lee [19] and Toledo-Tong [26]. Under the condition (1.10), in [21] we constructed a new Lagrangian cycle $LC(F, \Phi)_M$ in the cotangent bundle T^*M . We call it the Lefschetz cycle associated with the pair (F, Φ) and the fixed point component M . Note that in the more general setting of elliptic pairs a similar construction of microlocal Lefschetz classes was previously given in Guillermou [12]. The difference from his is that we explicitly realized them as Lagrangian cycles in T^*M . For recent results on this subject, see also [14], [18] and [23] etc. Note also that if $\phi = \text{id}_X$, $M = X$ and $\Phi = \text{id}_F$, our Lefschetz cycle $LC(F, \Phi)_M$ coincides with the characteristic cycle $CC(F)$ of F introduced by Kashiwara [15]. By Lefschetz cycles, in [21] we could generalize almost all nice properties of characteristic cycles. In particular, we have the following microlocal index theorem for the local contribution $c(F, \Phi)_M$ from M .

Theorem 1.1 ([21, Theorem 4.8]) *Assume that $\text{supp}(F) \cap M$ is compact. Then for any continuous section $\sigma: M \longrightarrow T^*M$ of T^*M , we have*

$$c(F, \Phi)_M = \sharp([\sigma] \cap LC(F, \Phi)_M), \quad (1.11)$$

where $\sharp([\sigma] \cap LC(F, \Phi)_M)$ is the intersection number of the image of σ and $LC(F, \Phi)_M$ in the cotangent bundle T^*M .

However in [21] we could not describe $LC(F, \Phi)_M$ explicitly in terms of (F, Φ) and M since it was defined merely in an abstract manner by some morphisms in derived categories. Our answer to this problem is as follows. Let

$$CC: \text{CF}(M)_{\mathbb{C}} \xrightarrow{\sim} \Gamma(T^*M; \mathcal{L}_M) \quad (1.12)$$

be the isomorphism between the \mathbb{C} -vector space consisting of the \mathbb{C} -valued (subanalytically) constructible functions on M and that of closed conic subanalytic Lagrangian cycles on T^*M with coefficients in \mathbb{C} (see Proposition 2.10). Then in Section 5 we define a \mathbb{C} -valued constructible function $\theta(F, \Phi)_M \in \text{CF}(M)_{\mathbb{C}}$ on M associated to the hyperbolic localization of the specialization $\nu_M(F)$ of F in the sense of Braden [1], Braden-Proudfoot [2] and Goresky-MacPherson [9]. More precisely, for each point $x \in M$ by taking an expanding subbundle (see Definition 5.3) $\mathcal{E} \subset \mathcal{G}|_U$ of $\mathcal{G} = T_M X$ on its neighborhood $U \subset M$ we set

$$\theta(F, \Phi)_M(x) := \sum_{j \in \mathbb{Z}} (-1)^j \text{tr} \{ H^j(\nu_M(F)_{\mathcal{E}}^{\dagger-1})_x \xrightarrow{(\Phi')_{\mathcal{E}}^{\dagger-1}|_{\{x\}}} H^j(\nu_M(F)_{\mathcal{E}}^{\dagger-1})_x \}, \quad (1.13)$$

where $\nu_M(F)_{\mathcal{E}}^{\dagger-1} \in \mathbf{D}_{\mathbb{R}-c}^b(U)$ is the hyperbolic localization of $\nu_M(F)$ with respect to \mathcal{E} and $(\Phi')_{\mathcal{E}}^{\dagger-1}: \nu_M(F)_{\mathcal{E}}^{\dagger-1} \longrightarrow \nu_M(F)_{\mathcal{E}}^{\dagger-1}$ is its endomorphism induced by Φ . Then we have the following result.

Theorem 1.2 *We have the equality*

$$LC(F, \Phi)_M = CC(\theta(F, \Phi)_M) \quad (1.14)$$

*as Lagrangian cycles in T^*M . In particular, if moreover $\text{supp}(F) \cap M$ is compact we have*

$$c(F, \Phi)_M = \int_M \theta(F, \Phi)_M, \quad (1.15)$$

where $\int_M : \text{CF}(M)_{\mathbb{C}} \rightarrow \mathbb{C}$ is the morphism defined by topological (Euler) integrals (see Definition 2.9).

Note that when $\dim M = 0$ this theorem coincides with Kashiwara's one in [16, Proposition 1.4.1] (see also Kashiwara-Schapira [17, Proposition 9.6.12]). It gives also an affirmative answer to a conjecture of Goresky-MacPherson [8] in particular for smooth fixed point components (see also [9, page 9, (1.12) Open problems]). In the previous results in [9], the authors assumed a technical condition that there exists a special indicator map $t : W \rightarrow \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$ on a neighborhood W of M in X (see [9, Definition (3.1)]). They call such an endomorphism ϕ a weakly hyperbolic map. Moreover by taking \mathcal{E} to be the minimal expanding subbundle (see Definition 5.2) $\mathcal{W} \subset \mathcal{G}|_U$ of $\mathcal{G} = T_M X$ on $U \subset M$ we can reduce the calculation of the local contribution $c(F, \Phi)_M$ to that on a subset $\mathcal{W} \subset \mathcal{G}|_U$ much smaller than the one in [9]. In contrast to loc. cit., at each point of M we can take the smallest possible \mathcal{E} containing only the generalized eigenspaces for the eigenvalues of ϕ'_x in $\mathbb{R}_{>1}$. Compare our stalk formula of $\nu_M(F)_{\mathcal{E}}^{!-1}$ in Proposition 5.13 with the theorem in [9, page 6]. Thus Theorem 1.2 improves the results in [16], [17] and [9] etc. In order to prove it even in the difficult case where the set $\text{Ev}(\phi'_x)$ varies depending on $x \in \text{supp}(F) \cap M$, we require some precise arguments on our Lefschetz cycles (see Remark 2.16). More precisely, we encode Kashiwara's characteristic class $C(F, \Phi)_M \in H_{\text{supp}(F) \cap M}^n(X; or_X)$ satisfying $\int_X C(F, \Phi)_M = c(F, \Phi)_M$ to the geometric cycle $LC(F, \Phi)_M$ in T^*M and calculate the latter locally by applying our microlocal index theorem to hyperbolic localizations. Namely $LC(F, \Phi)_M$ enables us to patch local calculations to a global one. See Section 5 for the details. Finally in Section 6 we will show various examples for which the function $\theta(F, \Phi)_M$ and the local contribution $c(F, \Phi)_M \in \mathbb{C}$ can be explicitly calculated. In particular, we will give a very short proof to the following result proved first in [21] by using some deep results on the functorial properties of Lefschetz cycles (see [21, Sections 5 and 6]). Let $M = \sqcup_{\alpha \in A} M_{\alpha}$ be the decomposition of $M = (M_i)_{\text{reg}}$ into its connected components. Denote the sign of the determinant of the linear map

$$\text{id} - \phi' : T_{M_{\alpha}} X \rightarrow T_{M_{\alpha}} X \quad (1.16)$$

simply by $\text{sgn}(\text{id} - \phi')_{M_{\alpha}} \in \{\pm 1\}$.

Theorem 1.3 ([21, Corollary 6.5]) *In addition to the condition (1.10), assume that the inclusion map $i_M : M \hookrightarrow X$ is non-characteristic for F and $\text{supp}(F) \cap M$ is compact. Then we have*

$$c(F, \Phi)_M = \sum_{\alpha \in A} \text{sgn}(\text{id} - \phi')_{M_{\alpha}} \cdot \int_{M_{\alpha}} \varphi(F|_{M_{\alpha}}, \Phi|_{M_{\alpha}}), \quad (1.17)$$

where the \mathbb{C} -valued constructible function $\varphi(F|_{M_\alpha}, \Phi|_{M_\alpha}) \in \text{CF}(M_\alpha)_{\mathbb{C}}$ on M_α is defined by

$$\varphi(F|_{M_\alpha}, \Phi|_{M_\alpha})(x) := \sum_{j \in \mathbb{Z}} (-1)^j \text{tr}\{H^j(F)_x \xrightarrow{\Phi|_{\{x\}}} H^j(F)_x\} \quad (1.18)$$

for $x \in M_\alpha$.

Thus Theorems 1.2 and 1.3 show that we can calculate the global trace of (F, Φ) locally at each point of M without assuming any further technical condition such as

$$\text{Ev}(\phi'_x) \cap \mathbb{R}_{>1} = \emptyset \quad \text{for any } x \in M \quad (1.19)$$

on the map ϕ . Note that if there exists a point $x \in M$ such that $\text{Ev}(\phi'_x) \cap \mathbb{R}_{>1} \neq \emptyset$ the classical methods (see e.g. [17, Section 9.6]) for localizations do not work. Our main result in this paper, that is Theorem 1.2 is much more general than Theorem 1.3 since in the former we do not assume that $i_M: M \hookrightarrow X$ is non-characteristic for F . In particular, it immediately implies general Lefschetz fixed point formulas for singular subvarieties $V \subset X$ of X such that $\phi(V) \subset V$ by applying it to the special case where $F = \mathbb{C}_V$ and Φ is the natural morphism $\phi^{-1}\mathbb{C}_V = \mathbb{C}_{\phi^{-1}(V)} \longrightarrow \mathbb{C}_V$. Note that a complete answer to this problem is known only for some special cases (for the case of normal complex algebraic surfaces, see Saito [24]).

2 Preliminary notions and results

In this paper, we essentially follow the terminology in [13] and [17]. For example, for a topological space X , we denote by $\mathbf{D}^b(X)$ the derived category of bounded complexes of sheaves of \mathbb{C}_X -modules on X . From now on, we shall review basic notions and known results concerning Lefschetz fixed point formulas. Since we focus our attention on Lefschetz fixed point formulas for constructible sheaves in this paper, we treat here only real analytic manifolds and morphisms. Now let X be a real analytic manifold. We denote by $\mathbf{D}_{\mathbb{R}-c}^b(X)$ the full subcategory of $\mathbf{D}^b(X)$ consisting of bounded complexes of sheaves whose cohomology sheaves are \mathbb{R} -constructible (see [17, Chapter VIII] for the precise definition). Let $\phi: X \longrightarrow X$ be an endomorphism of the real analytic manifold X . Then our initial datum is a pair (F, Φ) of $F \in \mathbf{D}_{\mathbb{R}-c}^b(X)$ and a morphism $\Phi: \phi^{-1}F \longrightarrow F$ in $\mathbf{D}_{\mathbb{R}-c}^b(X)$. If the support $\text{supp}(F)$ of F is compact, $H^j(X; F)$ is a finite-dimensional vector space over \mathbb{C} for any $j \in \mathbb{Z}$ and we can define the following important number from (F, Φ) .

Definition 2.1 We set

$$\text{tr}(F, \Phi) := \sum_{j \in \mathbb{Z}} (-1)^j \text{tr}\{H^j(X; F) \xrightarrow{\Phi} H^j(X; F)\} \in \mathbb{C}, \quad (2.1)$$

where the morphisms $H^j(X; F) \xrightarrow{\Phi} H^j(X; F)$ are induced by

$$F \longrightarrow R\phi_*\phi^{-1}F \xrightarrow{\Phi} R\phi_*F. \quad (2.2)$$

We call $\text{tr}(F, \Phi)$ the global trace of the pair (F, Φ) .

Now let us set

$$M := \{x \in X \mid \phi(x) = x\} \subset X. \quad (2.3)$$

This is the fixed point set of $\phi: X \rightarrow X$ in X . Since we mainly consider the case where the fixed point set is a smooth submanifold of X , we use the symbol M to express it. If a compact group G is acting on X and ϕ is the left action of an element of G , then the fixed point set is smooth by Palais's theorem [22] (see [11] for an excellent survey of this subject). Now let us consider the diagonal embedding $\delta_X: X \hookrightarrow X \times X$ of X and the closed embedding $h := (\phi, \text{id}_X): X \hookrightarrow X \times X$ associated with ϕ . Denote by Δ_X (resp. Γ_ϕ) the image of X by δ_X (resp. h). Then $M \simeq \Delta_X \cap \Gamma_\phi$ and we obtain a chain of morphisms

$$R\mathcal{H}om_{\mathbb{C}_X}(F, F) \simeq \delta_X^!(F \boxtimes DF) \quad (2.4)$$

$$\longrightarrow R\Gamma_{\text{supp}(F) \cap \Delta_X}(h_* h^{-1}(F \boxtimes DF))|_{\Delta_X} \quad (2.5)$$

$$\simeq R\Gamma_{\text{supp}(F) \cap \Delta_X}(h_*(\phi^{-1}F \otimes DF))|_{\Delta_X} \quad (2.6)$$

$$\xrightarrow{\Phi} R\Gamma_{\text{supp}(F) \cap \Delta_X}(h_*(F \otimes DF))|_{\Delta_X} \quad (2.7)$$

$$\longrightarrow R\Gamma_{\text{supp}(F) \cap \Delta_X}(h_* \omega_X)|_{\Delta_X} \quad (2.8)$$

$$\simeq R\Gamma_{\text{supp}(F) \cap M}(\omega_X), \quad (2.9)$$

where $\omega_X \simeq or_X[\dim X] \in \mathbf{D}_{\mathbb{R}-c}^b(X)$ is the dualizing complex of X and $DF = R\mathcal{H}om_{\mathbb{C}_X}(F, \omega_X)$ is the Verdier dual of F . Hence we get a morphism

$$\text{Hom}_{\mathbf{D}^b(X)}(F, F) \longrightarrow H_{\text{supp}(F) \cap M}^0(X; \omega_X). \quad (2.10)$$

Definition 2.2 ([16]) We denote by $C(F, \Phi)$ the image of id_F by the morphism (2.10) in $H_{\text{supp}(F) \cap M}^0(X; \omega_X)$ and call it the characteristic class of (F, Φ) .

Theorem 2.3 ([16]) *If $\text{supp}(F)$ is compact, then the equality*

$$\text{tr}(F, \Phi) = \int_X C(F, \Phi) \quad (2.11)$$

holds. Here

$$\int_X : H_c^n(X; or_X) \longrightarrow \mathbb{C} \quad (2.12)$$

is the morphism induced by the integral of differential $(\dim X)$ -forms with compact support.

Let $M = \bigsqcup_{i \in I} M_i$ be the decomposition of M into connected components and

$$H_{\text{supp}(F) \cap M}^0(X; \omega_X) = \bigoplus_{i \in I} H_{\text{supp}(F) \cap M_i}^0(X; \omega_X), \quad (2.13)$$

$$C(F, \Phi) = \bigoplus_{i \in I} C(F, \Phi)_{M_i} \quad (2.14)$$

the associated direct sum decomposition.

Definition 2.4 When $\text{supp}(F) \cap M_i$ is compact, we define a complex number $c(F, \Phi)_{M_i}$ by

$$c(F, \Phi)_{M_i} := \int_X C(F, \Phi)_{M_i} \quad (2.15)$$

and call it the local contribution of (F, Φ) from M_i .

By Theorem 2.3, if $\text{supp}(F)$ is compact, the global trace of (F, Φ) is the sum of local contributions:

$$\text{tr}(F, \Phi) = \sum_{i \in I} c(F, \Phi)_{M_i}. \quad (2.16)$$

Hence one of the most important problems in the theory of Lefschetz fixed point formulas is to explicitly describe these local contributions. However the direct computation of local contributions is a very difficult task in general. Instead of local contributions, we usually consider first the following number $\text{tr}(F|_{M_i}, \Phi|_{M_i})$ which is much more easily computed. Let M_i be a fixed point component such that $\text{supp}(F) \cap M_i$ is compact.

Definition 2.5 We set

$$\text{tr}(F|_{M_i}, \Phi|_{M_i}) := \sum_{j \in \mathbb{Z}} (-1)^j \text{tr}\{H^j(M_i; F|_{M_i}) \xrightarrow{\Phi|_{M_i}} H^j(M_i; F|_{M_i})\}, \quad (2.17)$$

where the morphisms $H^j(M_i; F|_{M_i}) \xrightarrow{\Phi|_{M_i}} H^j(M_i; F|_{M_i})$ are induced by the restriction

$$\Phi|_{M_i}: F|_{M_i} \simeq (\phi^{-1}F)|_{M_i} \longrightarrow F|_{M_i} \quad (2.18)$$

of Φ .

We can easily compute this new invariant $\text{tr}(F|_{M_i}, \Phi|_{M_i}) \in \mathbb{C}$ as follows. Let $M_i = \bigsqcup_{\alpha \in A} M_{i,\alpha}$ be a stratification of M_i by connected subanalytic manifolds $M_{i,\alpha}$ such that $H^j(F)|_{M_{i,\alpha}}$ is a locally constant sheaf for any $\alpha \in A$ and $j \in \mathbb{Z}$. Namely, we assume that the stratification $M_i = \bigsqcup_{\alpha \in A} M_{i,\alpha}$ is adapted to $F|_{M_i}$.

Definition 2.6 For each $\alpha \in A$, we set

$$c_\alpha := \sum_{j \in \mathbb{Z}} (-1)^j \text{tr}\{H^j(F)_{x_\alpha} \xrightarrow{\Phi|_{\{x_\alpha\}}} H^j(F)_{x_\alpha}\} \in \mathbb{C}, \quad (2.19)$$

where x_α is a reference point of $M_{i,\alpha}$.

Then we have the following very useful result due to Goresky-MacPherson.

Proposition 2.7 ([9]) *In the situation as above, we have*

$$\text{tr}(F|_{M_i}, \Phi|_{M_i}) = \sum_{\alpha \in A} c_\alpha \cdot \chi_c(M_{i,\alpha}), \quad (2.20)$$

where χ_c is the Euler-Poincaré index with compact supports.

In terms of the theory of topological integrals of constructible functions developed by Kashiwara-Schapira [17], Schapira [25] and Viro [27] etc., we can restate this result in the following way. Since we need \mathbb{C} -valued constructible functions, we slightly generalize the usual notion of \mathbb{Z} -valued constructible functions.

Definition 2.8 Let Z be a subanalytic set. Then we say that a \mathbb{C} -valued function $\varphi: Z \rightarrow \mathbb{C}$ is constructible if there exists a stratification $Z = \bigsqcup_{\alpha \in A} Z_\alpha$ of Z by subanalytic manifolds Z_α such that $\varphi|_{Z_\alpha}$ is a constant function for any $\alpha \in A$. We denote by $\text{CF}(Z)_{\mathbb{C}}$ the abelian group of \mathbb{C} -valued constructible functions on Z .

Let $\varphi = \sum_{\alpha \in A} c_\alpha \cdot \mathbf{1}_{Z_\alpha} \in \text{CF}(Z)_{\mathbb{C}}$ be a \mathbb{C} -valued constructible function with compact support on a subanalytic set Z , where $Z = \bigsqcup_{\alpha \in A} Z_\alpha$ is a stratification of Z and $c_\alpha \in \mathbb{C}$. Then we can easily prove that the complex number $\sum_{\alpha \in A} c_\alpha \cdot \chi_c(Z_\alpha)$ does not depend on the expression $\varphi = \sum_{\alpha \in A} c_\alpha \cdot \mathbf{1}_{Z_\alpha}$ of φ .

Definition 2.9 For a \mathbb{C} -valued constructible function $\varphi = \sum_{\alpha \in A} c_\alpha \cdot \mathbf{1}_{Z_\alpha} \in \text{CF}(Z)_{\mathbb{C}}$ with compact support as above, we set

$$\int_Z \varphi := \sum_{\alpha \in A} c_\alpha \cdot \chi_c(Z_\alpha) \in \mathbb{C} \quad (2.21)$$

and call it the topological integral of φ .

By this definition, the result of Proposition 2.7 can be rewritten as

$$\text{tr}(F|_{M_i}, \Phi|_{M_i}) = \int_{M_i} \varphi(F|_{M_i}, \Phi|_{M_i}), \quad (2.22)$$

where the \mathbb{C} -valued constructible function $\varphi(F|_{M_i}, \Phi|_{M_i}) \in \text{CF}(M_i)_{\mathbb{C}}$ on M_i is defined by

$$\varphi(F|_{M_i}, \Phi|_{M_i})(x) := \sum_{j \in \mathbb{Z}} (-1)^j \text{tr} \{ H^j(F)_x \xrightarrow{\Phi|_{\{x\}}} H^j(F)_x \} \quad (2.23)$$

for $x \in M_i$.

Let us explain how the \mathbb{C} -valued constructible functions discussed above are related to the theory of Lagrangian cycles in [17, Chapter IX]. Now let Z be a real analytic manifold and denote by T^*Z its cotangent bundle. Recall that Kashiwara-Schapira constructed the sheaf \mathcal{L}_Z of closed conic subanalytic Lagrangian cycles on T^*Z in [17] (in this paper, we consider Lagrangian cycles with coefficients in \mathbb{C}).

Proposition 2.10 ([17]) *There exists a group isomorphism*

$$CC: \text{CF}(Z)_{\mathbb{C}} \xrightarrow{\sim} \Gamma(T^*Z; \mathcal{L}_Z) \quad (2.24)$$

*by which the characteristic function $\mathbf{1}_K$ of a closed submanifold $K \subset Z$ of Z is sent to the conormal cycle $[T_K^*Z]$ in T^*Z .*

We call CC the characteristic cycle map in this paper. From now on, we fix a fixed point component M_i and always assume that $\text{supp}(F) \cap M_i$ is compact.

Definition 2.11 We say that the global trace $\text{tr}(F, \Phi)$ is localizable to M_i if the equality

$$c(F, \Phi)_{M_i} = \text{tr}(F|_{M_i}, \Phi|_{M_i}) \quad (2.25)$$

holds.

By Proposition 2.7, once the global trace is localizable to M_i , the local contribution $c(F, \Phi)_{M_i}$ of (F, Φ) from M_i can be very easily computed. Let us denote M_i , $c(F, \Phi)_{M_i}$ etc. simply by M , $c(F, \Phi)_M$ etc. respectively. From now on, we shall introduce some useful criterions for the localizability of the global trace to M . First let us consider the natural morphism

$$\phi': T_{M_{\text{reg}}} X \longrightarrow T_{M_{\text{reg}}} X \quad (2.26)$$

induced by $\phi: X \longrightarrow X$, where M_{reg} denotes the set of regular points in M . Since M_{reg} is not always connected in the real analytic case, the rank of $T_{M_{\text{reg}}} X$ may vary depending on the connected components of M_{reg} .

Definition 2.12 Let V be a finite-dimensional vector space over \mathbb{R} . For its \mathbb{R} -linear endomorphism $A: V \longrightarrow V$, we set

$$\text{Ev}(A) := \{\text{the eigenvalues of } A^{\mathbb{C}}: V^{\mathbb{C}} \longrightarrow V^{\mathbb{C}}\} \subset \mathbb{C}, \quad (2.27)$$

where $V^{\mathbb{C}}$ is the complexification of V .

In particular, for $x \in M_{\text{reg}}$ we set

$$\text{Ev}(\phi'_x) := \{\text{the eigenvalues of } \phi'_x: (T_{M_{\text{reg}}} X)_x \longrightarrow (T_{M_{\text{reg}}} X)_x\} \subset \mathbb{C}. \quad (2.28)$$

We also need the specialization functor

$$\nu_{M_{\text{reg}}}: \mathbf{D}^b(X) \longrightarrow \mathbf{D}^b(T_{M_{\text{reg}}} X) \quad (2.29)$$

along $M_{\text{reg}} \subset X$. In order to recall the construction of this functor, consider the standard commutative diagram:

$$\begin{array}{ccc} T_{M_{\text{reg}}} X & \xrightarrow{s} & \widetilde{X_{M_{\text{reg}}}} \xleftarrow{j} \Omega_X \\ \downarrow \tau & & \downarrow p \swarrow \tilde{p} \\ M_{\text{reg}} & \xrightarrow{i} & X, \end{array} \quad (2.30)$$

where $\widetilde{X_{M_{\text{reg}}}}$ is the normal deformation of X along M_{reg} and $t: \widetilde{X_{M_{\text{reg}}}} \longrightarrow \mathbb{R}$ is the deformation parameter. Recall that Ω_X is defined by $t > 0$ in $\widetilde{X_{M_{\text{reg}}}}$. Then the specialization $\nu_{M_{\text{reg}}}(F)$ of F along M_{reg} is defined by

$$\nu_{M_{\text{reg}}}(F) := s^{-1} Rj_* \tilde{p}^{-1}(F). \quad (2.31)$$

Note that $\nu_{M_{\text{reg}}}(F)$ is a conic object in $\mathbf{D}^b(T_{M_{\text{reg}}} X)$ whose support is contained in the normal cone $C_{M_{\text{reg}}}(\text{supp}(F))$ to $\text{supp}(F)$ along M_{reg} . Since F is \mathbb{R} -constructible, $\nu_{M_{\text{reg}}}(F)$ is also \mathbb{R} -constructible. By construction, there exists a natural morphism

$$\Phi': (\phi')^{-1} \nu_{M_{\text{reg}}}(F) \longrightarrow \nu_{M_{\text{reg}}}(F) \quad (2.32)$$

induced by $\Phi: \phi^{-1} F \longrightarrow F$. In the sequel, let us assume the conditions:

- (i) $\text{supp}(F) \cap M$ is compact and contained in M_{reg} .
- (ii) $1 \notin \text{Ev}(\phi'_x)$ for any $x \in \text{supp}(F) \cap M_{\text{reg}}$.

The condition (ii) implies that the graph of ϕ in $X \times X$ intersects cleanly (see [17, Definition 4.1.5]) with the diagonal set $\Delta_X \simeq X$ in an open neighborhood of $\text{supp}(F) \cap M_{\text{reg}}$. It follows also from the condition (ii) that for an open neighborhood U of $\text{supp}(F) \cap M_{\text{reg}}$ in M_{reg} the fixed point set of $\phi'|_{\tau^{-1}(U)}: \tau^{-1}(U) \rightarrow \tau^{-1}(U)$ is contained in the zero-section M_{reg} of $T_{M_{\text{reg}}}X$. Set $\tilde{U} = \tau^{-1}(U)$, $\tilde{F} = \nu_{M_{\text{reg}}}(F)|_{\tilde{U}}$ and $\tilde{\Phi} = \Phi'|_{\tilde{U}}: (\phi'|_{\tilde{U}})^{-1}\tilde{F} \rightarrow \tilde{F}$. Then also for the pair $(\tilde{F}, \tilde{\Phi})$, we can define the characteristic class $C(\tilde{F}, \tilde{\Phi}) \in H^0_{\text{supp}(F) \cap M_{\text{reg}}}(\tilde{U}; \omega_{\tilde{U}})$.

Proposition 2.13 ([21, Proposition 3.1]) *Under the conditions (i) and (ii), the local contribution $c(F, \Phi)_M$ from M is equal to $\int_{\tilde{U}} C(\tilde{F}, \tilde{\Phi})$.*

In [21, Theorem 3.2] we proved the following result by Proposition 2.13.

Theorem 2.14 ([21, Theorem 3.2]) *Under the conditions (i) and (ii), assume moreover that*

$$\text{Ev}(\phi'_x) \cap \mathbb{R}_{>1} = \emptyset \quad (2.33)$$

for any $x \in \text{supp}(F) \cap M \subset M_{\text{reg}}$. Then the localization

$$c(F, \Phi)_M = \text{tr}(F|_M, \Phi|_M) = \int_M \varphi(F|_M, \Phi|_M) \quad (2.34)$$

holds.

In the complex case, we have the following stronger result.

Theorem 2.15 ([21, Theorem 3.3]) *Under the conditions (i) and (ii), assume moreover that X and $\phi: X \rightarrow X$ are complex analytic and $F \in \mathbf{D}_c^b(X)$ i.e. F is \mathbb{C} -constructible. Assume also that there exists a compact complex manifold N such that $\text{supp}(F) \cap M \subset N \subset M$. Then the localization*

$$c(F, \Phi)_M = \text{tr}(F|_M, \Phi|_M) = \int_M \varphi(F|_M, \Phi|_M) \quad (2.35)$$

holds.

Remark 2.16 Later we will generalize Theorems 2.14 and 2.15. To treat the more general case where the set $\text{Ev}(\phi'_x)$ may vary depending on $x \in \text{supp}(F) \cap M$, we need some precise arguments on Lefschetz cycles which will be introduced in the next section. One naive idea to treat this case would be to cover $\text{supp}(F) \cap M$ by sufficiently small closed subsets $Z_i \subset \text{supp}(F) \cap M$ and use the local contributions of $(\nu_{M_{\text{reg}}}(F))_{\tau^{-1}Z_i}$ to compute that of $\nu_{M_{\text{reg}}}(F)$ by a Mayer-Vietoris type argument. However this very simple idea does not work, because we cannot apply [17, Proposition 9.6.2] to constructible sheaves with “non-compact” support such as $(\nu_{M_{\text{reg}}}(F))_{\tau^{-1}Z_i}$ to justify the Mayer-Vietoris type argument.

3 Some properties of Lefschetz cycles

In this section, we recall our construction of Lefschetz cycles in [21] and their standard properties. We inherit the notations in Section 2. Now assume that the fixed point set $M = \{x \in X \mid \phi(x) = x\}$ of $\phi: X \rightarrow X$ is a submanifold of X . However here we do not assume that M is connected. We also assume that Δ_X intersects with $\Gamma_\phi = \{(\phi(x), x) \in X \times X \mid x \in X\}$ cleanly along M in $X \times X$. Identifying Γ_ϕ with X by the second projection $X \times X \rightarrow X$, we obtain a natural identification $M = \Gamma_\phi \cap \Delta_X$. We also identify $T_{\Delta_X}^*(X \times X)$ with T^*X by the first projection $T^*(X \times X) \simeq T^*X \times T^*X \rightarrow T^*X$ as usual.

Lemma 3.1 ([21, Lemma 4.1]) *The subset $T_{\Gamma_\phi}^*(X \times X) \cap T_{\Delta_X}^*(X \times X)$ of $(\Gamma_\phi \cap \Delta_X) \times_{\Delta_X} T_{\Delta_X}^*(X \times X) \simeq M \times_X T^*X$ is a subbundle of $M \times_X T^*X$ (whose rank as a vector bundle may vary depending on the connected components of M).*

Definition 3.2 ([21, Definition 4.2]) We denote the subbundle $T_{\Gamma_\phi}^*(X \times X) \cap T_{\Delta_X}^*(X \times X)$ of $M \times_X T^*X$ by \mathcal{F} and call it the Lefschetz bundle associated with $\phi: X \rightarrow X$.

Proposition 3.3 ([21, Proposition 4.2]) *The natural surjective morphism $\rho: M \times_X T^*X \rightarrow T^*M$ induces an isomorphism $\mathcal{F} \xrightarrow{\sim} T^*M$.*

From now on, by Proposition 3.3 we shall identify the Lefschetz bundle \mathcal{F} with T^*M .

Now let F be an object of $\mathbf{D}_{\mathbb{R}-c}^b(X)$ and $\Phi: \phi^{-1}F \rightarrow F$ a morphism in $\mathbf{D}_{\mathbb{R}-c}^b(X)$. To these data (F, Φ) , we can associate a conic Lagrangian cycle in the Lefschetz bundle $\mathcal{F} \simeq T^*M$ as follows. Denote by $\pi_X: T^*X \rightarrow X$ the natural projection and recall that we have the functor

$$\mu_{\Delta_X}: \mathbf{D}^b(X \times X) \rightarrow \mathbf{D}^b(T_{\Delta_X}^*(X \times X)) \quad (3.1)$$

of microlocalization which satisfies

$$R\pi_{X*}\mu_{\Delta_X} \simeq \delta_X^! \simeq \delta_X^{-1}R\Gamma_{\Delta_X}. \quad (3.2)$$

Recall also that the micro-support $\text{SS}(F)$ of F is a closed conic subanalytic Lagrangian subset of T^*X and the support of $\mu_{\Delta_X}(F \boxtimes DF)$ is contained in $\text{SS}(F) \subset T^*X \simeq T_{\Delta_X}^*(X \times X)$. Then we have a chain of natural morphisms:

$$R\text{Hom}_{\mathbb{C}_X}(F, F) \simeq R\Gamma(X; \delta_X^!(F \boxtimes DF)) \quad (3.3)$$

$$\simeq R\Gamma_{\text{SS}(F)}(T^*X; \mu_{\Delta_X}(F \boxtimes DF)) \quad (3.4)$$

$$\rightarrow R\Gamma_{\text{SS}(F)}(T^*X; \mu_{\Delta_X}(h_*h^{-1}(F \boxtimes DF))) \quad (3.5)$$

$$\simeq R\Gamma_{\text{SS}(F)}(T^*X; \mu_{\Delta_X}(h_*(\phi^{-1}F \otimes DF))) \quad (3.6)$$

$$\xrightarrow{\Phi} R\Gamma_{\text{SS}(F)}(T^*X; \mu_{\Delta_X}(h_*(F \otimes DF))) \quad (3.7)$$

$$\rightarrow R\Gamma_{\text{SS}(F)}(T^*X; \mu_{\Delta_X}(h_*\omega_X)). \quad (3.8)$$

Lemma 3.4 ([21, Lemma 4.4])

- (i) *The support of $\mu_{\Delta_X}(h_*\omega_X)$ is contained in \mathcal{F} .*

- (ii) The restriction of $\mu_{\Delta_X}(h_*\omega_X)$ to $\mathcal{F} \simeq T^*M$ is isomorphic to $\pi_M^{-1}\omega_M$, where $\pi_M: T^*M \rightarrow M$ is the natural projection.

By Lemma 3.4 there exists an isomorphism

$$\mu_{\Delta_X}(h_*\omega_X) \simeq (i_{\mathcal{F}})_*\pi_M^{-1}\omega_M, \quad (3.9)$$

where $i_{\mathcal{F}}: \mathcal{F} \hookrightarrow T^*X$ is the inclusion map. In what follows, we sometimes omit the symbol $(i_{\mathcal{F}})_*$ in the above identification (3.9). Combining the chain of morphisms (3.3)-(3.8) with the isomorphism (3.9), we obtain a morphism

$$\mathrm{Hom}_{\mathbf{D}^b(X)}(F, F) \longrightarrow H_{\mathrm{SS}(F) \cap \mathcal{F}}^0(\mathcal{F}; \pi_M^{-1}\omega_M). \quad (3.10)$$

Definition 3.5 ([21, Definition 4.5]) We denote by $LC(F, \Phi)$ the image of $\mathrm{id}_F \in \mathrm{Hom}_{\mathbf{D}^b(X)}(F, F)$ in $H_{\mathrm{SS}(F) \cap \mathcal{F}}^0(\mathcal{F}; \pi_M^{-1}\omega_M)$ by the morphism (3.10).

Lemma 3.6 ([21, Lemma 4.6]) $\mathrm{SS}(F) \cap \mathcal{F}$ is contained in a closed conic subanalytic Lagrangian subset of $\mathcal{F} \simeq T^*M$.

Definition 3.7 ([21, Definition 4.7]) Choose a closed conic subanalytic Lagrangian subset Λ of $\mathcal{F} \simeq T^*M$ such that $\mathrm{SS}(F) \cap \mathcal{F} \subset \Lambda$. We consider $LC(F, \Phi)$ as an element of $H_{\Lambda}^0(\mathcal{F}; \pi_M^{-1}\omega_M)$ and call it the Lefschetz cycle associated with the pair (F, Φ) .

As a basic property of Lefschetz cycles, we have the following homotopy invariance. Let $I = [0, 1]$ and let $\phi: X \times I \rightarrow X$ be the restriction of a morphism of real analytic manifolds $X \times \mathbb{R} \rightarrow X$. For $t \in I$, let $i_t: X \hookrightarrow X \times I$ be the injection defined by $x \mapsto (x, t)$ and set $\phi_t := \phi \circ i_t: X \rightarrow X$. Assume that the fixed point set of ϕ_t in X is smooth and does not depend on $t \in I$. We denote this fixed point set by M . Let $F \in \mathbf{D}_{\mathbb{R}-c}^b(X)$ and consider a morphism $\Phi: \phi^{-1}F \rightarrow p^{-1}F$ in $\mathbf{D}_{\mathbb{R}-c}^b(X \times I)$, where $p: X \times I \rightarrow X$ is the projection. We set

$$\Phi_t := \Phi|_{X \times \{t\}}: \phi_t^{-1}F \rightarrow F \quad (3.11)$$

for $t \in I$. We denote the Lefschetz bundle associated with ϕ_t by $\mathcal{F}_t \simeq T^*M$.

Proposition 3.8 Assume that $\mathrm{supp}(F) \cap M$ is compact and $\mathrm{SS}(F) \cap \mathcal{F}_t \subset T^*M$ does not depend on $t \in I$ as a subset of T^*M . Then the Lefschetz cycle $LC(F, \Phi_t) \in H_{\mathrm{SS}(F) \cap \mathcal{F}_t}^0(T^*M; \pi_M^{-1}\omega_M)$ does not depend on $t \in I$.

Proof. The proof proceeds completely in the same way as that of [17, Proposition 9.6.8]. Hence we omit the detail. \square

4 Microlocal index formula for local contributions

In this section, using the Lefschetz cycle $LC(F, \Phi)$ introduced in Section 3, we introduce our microlocal index theorem which expresses local contributions of (F, Φ) as intersection numbers of the images of continuous sections of $\mathcal{F} \simeq T^*M$ and $LC(F, \Phi)$. Here we do not assume that the fixed point set M of $\phi: X \rightarrow X$ is smooth. However we assume the condition:

$$1 \notin \text{Ev}(\phi'_x) \quad \text{for any } x \in M_{\text{reg}}. \quad (4.1)$$

Also in this more general setting, we can define the Lefschetz bundle $\mathcal{F} \simeq T^*M_{\text{reg}}$ over M_{reg} and construct the Lefschetz cycle $LC(F, \Phi)$ in \mathcal{F} by using the methods in Section 3. Let $M = \bigsqcup_{i \in I} M_i$ be the decomposition of M into connected components. Denote $(M_i)_{\text{reg}}$ simply by N_i and set $\mathcal{F}_i := N_i \times_{M_{\text{reg}}} \mathcal{F}$. Then we get a decomposition $\mathcal{F} = \bigsqcup_{i \in I} \mathcal{F}_i \simeq \bigsqcup_{i \in I} T^*N_i$ of \mathcal{F} . By the direct sum decomposition

$$H_{\text{SS}(F) \cap \mathcal{F}}^0(\mathcal{F}; \pi_{M_{\text{reg}}}^{-1} \omega_{M_{\text{reg}}}) \simeq \bigoplus_{i \in I} H_{\text{SS}(F) \cap \mathcal{F}_i}^0(\mathcal{F}_i; \pi_{N_i}^{-1} \omega_{N_i}), \quad (4.2)$$

we obtain a decomposition

$$LC(F, \Phi) = \sum_{i \in I} LC(F, \Phi)_{M_i} \quad (4.3)$$

of $LC(F, \Phi)$, where $LC(F, \Phi)_{M_i} \in H_{\text{SS}(F) \cap \mathcal{F}_i}^0(\mathcal{F}_i; \pi_{N_i}^{-1} \omega_{N_i})$. Now let us fix a fixed point component M_i and assume that $\text{supp}(F) \cap M_i$ is compact and contained in $N_i = (M_i)_{\text{reg}}$. We shall show how the local contribution $c(F, \Phi)_{M_i} \in \mathbb{C}$ of (F, Φ) from M_i can be expressed by $LC(F, \Phi)_{M_i}$. In order to state our results, for the sake of simplicity, we denote $N_i = (M_i)_{\text{reg}}$, \mathcal{F}_i , $LC(F, \Phi)_{M_i}$, $c(F, \Phi)_{M_i}$ simply by M , \mathcal{F} , $LC(F, \Phi)$, $c(F, \Phi)$ respectively. Recall that to any continuous section $\sigma: M \rightarrow \mathcal{F} \simeq T^*M$ of the vector bundle \mathcal{F} , we can associate a cycle $[\sigma] \in H_{\sigma(M)}^0(T^*M; \pi_M^! \mathbb{C}_M)$ which is the image of $1 \in H^0(M; \mathbb{C}_M)$ by the isomorphism $H_{\sigma(M)}^0(T^*M; \pi_M^! \mathbb{C}_M) \simeq H^0(M; (\pi_M \circ \sigma)^! \mathbb{C}_M) \simeq H^0(M; \mathbb{C}_M)$ (see [17, Definition 9.3.5]). If $\sigma(M) \cap \text{supp}(LC(F, \Phi))$ is compact, we can define the intersection number $\sharp([\sigma] \cap LC(F, \Phi))$ of $[\sigma]$ and $LC(F, \Phi)$ to be the image of $[\sigma] \otimes LC(F, \Phi)$ by the chain of morphisms

$$H_{\sigma(M)}^0(\mathcal{F}; \pi_M^! \mathbb{C}_M) \otimes H_{\text{supp}(LC(F, \Phi))}^0(\mathcal{F}; \pi_M^{-1} \omega_M) \longrightarrow H_{\sigma(M) \cap \text{supp}(LC(F, \Phi))}^0(\mathcal{F}; \omega_{\mathcal{F}}) \quad (4.4)$$

$$\xrightarrow{\int_{\mathcal{F}}} \mathbb{C}. \quad (4.5)$$

Theorem 4.1 ([21, Theorem 4.8]) *Assume that $\text{supp}(F) \cap M$ is compact. Then for any continuous section $\sigma: M \rightarrow \mathcal{F} \simeq T^*M$ of \mathcal{F} , we have*

$$c(F, \Phi) = \sharp([\sigma] \cap LC(F, \Phi)). \quad (4.6)$$

As an application of Theorem 4.1, we shall give a useful formula which enables us to describe the Lefschetz cycle $LC(F, \Phi)$ explicitly in the special case where $\phi: X \rightarrow X$ is the identity map of X and $M = X$. For this purpose, until the end of this section, we shall consider the situation where $\phi = \text{id}_X$, $M = X$ and $\Phi: F \rightarrow F$ is an endomorphism of $F \in \mathbf{D}_{\mathbb{R}-c}^b(X)$. In this case, $LC(F, \Phi)$ is a Lagrangian cycle in T^*X . Now for real

analytic function $f: Y \rightarrow I$ on a real analytic manifold Y (I is an open interval in \mathbb{R}) we define a section $\sigma_f: Y \rightarrow T^*Y$ of T^*Y by $\sigma_f(y) := (y; df(y))$ ($y \in Y$) and set

$$\Lambda_f := \sigma_f(Y) = \{(y; df(y)) \mid y \in Y\}. \quad (4.7)$$

Note that Λ_f is a Lagrangian submanifold of T^*Y . Then we have the following analogue of [17, Theorem 9.5.3].

Theorem 4.2 *Let Y be a real analytic manifold, G an object of $\mathbf{D}_{\mathbb{R}-c}^b(Y)$ and $\Psi: G \rightarrow G$ an endomorphism of G . For a real analytic function $f: Y \rightarrow I$, assume that the following conditions are satisfied.*

- (i) $\text{supp}(G) \cap \{y \in Y \mid f(y) \leq t\}$ is compact for any $t \in I$.
- (ii) $\text{SS}(G) \cap \Lambda_f$ is compact.

Then the global trace

$$\text{tr}(G, \Psi) = \sum_{j \in \mathbb{Z}} (-1)^j \text{tr}\{H^j(Y; G) \xrightarrow{\Psi} H^j(Y; G)\} \quad (4.8)$$

of (G, Ψ) is equal to $\sharp([\sigma_f] \cap LC(G, \Psi))$.

Proof. Since the fixed point set of $\phi = \text{id}_Y$ is Y itself, $LC(G, \Psi)$ is a Lagrangian cycle in T^*Y . Moreover, since any open subset of Y is invariant by $\phi = \text{id}_Y$, we can freely use the microlocal Morse lemma ([17, Corollary 5.4.19]) to reduce the computation of the global trace $\text{tr}(G, \Psi)$ on Y to that of

$$\sum_{j \in \mathbb{Z}} (-1)^j \text{tr}\{H^j(\Omega_t; G) \xrightarrow{\Psi|_{\Omega_t}} H^j(\Omega_t; G)\} \quad (4.9)$$

for sufficiently large $t > 0$ in I , where we set $\Omega_t := \{y \in Y \mid f(y) < t\}$. Then the proof proceeds essentially in the same way as that of [17, Theorem 9.5.3]. \square

Theorem 4.3 *Let $X, F \in \mathbf{D}_{\mathbb{R}-c}^b(X)$ and $\Phi: F \rightarrow F$ be as above. For a real analytic function $f: X \rightarrow \mathbb{R}$ and a point $x_0 \in X$, assume the condition*

$$\Lambda_f \cap \text{SS}(F) \subset \{(x_0; df(x_0))\}. \quad (4.10)$$

Then the intersection number $\sharp([\sigma_f] \cap LC(F, \Phi))$ (at the point $(x_0; df(x_0)) \in T^*X$) is equal to

$$\sum_{j \in \mathbb{Z}} (-1)^j \text{tr}\{H_{\{f \geq f(x_0)\}}^j(F)_{x_0} \xrightarrow{\Phi} H_{\{f \geq f(x_0)\}}^j(F)_{x_0}\}. \quad (4.11)$$

Proof. The proof is very similar to that of [17, Theorem 9.5.6]. For a sufficiently small open ball $B(x_0, \varepsilon) = \{x \in X \mid |x - x_0| < \varepsilon\}$ centered at x_0 , set $F_0 = R\Gamma_{B(x_0, \varepsilon)}(F) \in \mathbf{D}_{\mathbb{R}-c}^b(X)$. Then Φ induces a natural morphism $\Phi_0: F_0 \rightarrow F_0$ in $\mathbf{D}_{\mathbb{R}-c}^b(X)$. Moreover by the proof of [17, Theorem 9.5.6], we have

$$\Lambda_f \cap \text{SS}(F_0) \subset \pi_X^{-1}(\Omega_{-t}) \sqcup \{(x_0; df(x_0))\} \quad (4.12)$$

for sufficiently small $t > 0$, where we set $\Omega_k := \{x \in X \mid f(x) - f(x_0) < k\}$ for $k \in \mathbb{R}$. Then applying Theorem 4.2 to the case where $I = (-\infty, 0)$, $Y = \Omega_0$, $G = F_0|_{\Omega_0} \in \mathbf{D}_{\mathbb{R}-c}^b(Y)$ and $\Psi = \Phi_0|_{\Omega_0}: G \longrightarrow G$, we obtain

$$\begin{aligned} & \sharp([\sigma_f] \cap LC(F_0, \Phi_0) \cap \pi_X^{-1}(\Omega_0)) \\ &= \sum_{j \in \mathbb{Z}} (-1)^j \text{tr} \{ H^j(B(x_0, \varepsilon) \cap \Omega_0; F) \xrightarrow{\Phi} H^j(B(x_0, \varepsilon) \cap \Omega_0; F) \}. \end{aligned} \quad (4.13)$$

On the other hand, since $\text{supp}(F_0)$ is compact in X , by Theorem 4.1 we have

$$\sharp([\sigma_f] \cap LC(F_0, \Phi_0)) = \sum_{j \in \mathbb{Z}} (-1)^j \text{tr} \{ H^j(B(x_0, \varepsilon); F) \xrightarrow{\Phi} H^j(B(x_0, \varepsilon); F) \}. \quad (4.14)$$

Comparing (4.13) with (4.14) in view of (4.12), we see that the intersection number of $[\sigma_f]$ and $LC(F_0, \Phi_0)$ at $(x_0; df(x_0))$ is equal to

$$\sum_{j \in \mathbb{Z}} (-1)^j \text{tr} \{ H_{\{f \geq f(x_0)\}}^j(F)_{x_0} \xrightarrow{\Phi} H_{\{f \geq f(x_0)\}}^j(F)_{x_0} \}. \quad (4.15)$$

Since $LC(F, \Phi) = LC(F_0, \Phi_0)$ in an open neighborhood of $(x_0; df(x_0))$ in T^*X , this last intersection number $\sharp([\sigma_f] \cap LC(F_0, \Phi_0))$ ($= (4.15)$) is equal to $\sharp([\sigma_f] \cap LC(F, \Phi))$. This completes the proof. \square

By Theorem 4.3, we can explicitly describe the Lefschetz cycle $LC(F, \Phi) \in \Gamma(T^*X; \mathcal{L}_X)$ as follows. Let $X = \bigsqcup_{\alpha \in A} X_\alpha$ be a μ -stratification of X such that

$$\text{supp}(LC(F, \Phi)) \subset \text{SS}(F) \subset \bigsqcup_{\alpha \in A} T_{X_\alpha}^* X. \quad (4.16)$$

Then $\Lambda := \bigsqcup_{\alpha \in A} T_{X_\alpha}^* X$ is a closed conic subanalytic Lagrangian subset of T^*X . Moreover there exists an open dense smooth subanalytic subset Λ_0 of Λ whose decomposition $\Lambda_0 = \bigsqcup_{i \in I} \Lambda_i$ into connected components satisfies the condition

$$\text{“For any } i \in I, \text{ there exists } \alpha_i \in A \text{ such that } \Lambda_i \subset T_{X_{\alpha_i}}^* X. \text{”} \quad (4.17)$$

Definition 4.4 For $i \in I$ and $\alpha_i \in A$ as above, we define a complex number $m_i \in \mathbb{C}$ by

$$m_i := \sum_{j \in \mathbb{Z}} (-1)^j \text{tr} \{ H_{\{f \geq f(x)\}}^j(F)_x \xrightarrow{\Phi} H_{\{f \geq f(x)\}}^j(F)_x \}, \quad (4.18)$$

where the point $x \in \pi_X(\Lambda_i) \subset X_{\alpha_i}$ and the \mathbb{R} -valued real analytic function $f: X \longrightarrow \mathbb{R}$ (defined in an open neighborhood of x in X) are defined as follows. Take a point $p \in \Lambda_i$ and set $x = \pi_X(p) \in X_{\alpha_i}$. Then $f: X \longrightarrow \mathbb{R}$ is a real analytic function which satisfies the following conditions:

- (i) $p = (x; df(x)) \in \Lambda_i$.
- (ii) The Hessian $\text{Hess}(f|_{X_{\alpha_i}})$ of $f|_{X_{\alpha_i}}$ is positive definite.

Corollary 4.5 *In the situation as above, for any $i \in I$ there exists an open neighborhood U_i of Λ_i in T^*X such that*

$$LC(F, \Phi) = m_i \cdot [T_{X_{\alpha_i}}^* X] \quad (4.19)$$

in U_i .

Now let us define a \mathbb{C} -valued constructible function $\varphi(F, \Phi)$ on X by

$$\varphi(F, \Phi)(x) := \sum_{j \in \mathbb{Z}} (-1)^j \operatorname{tr} \{ H^j(F)_x \xrightarrow{\Phi|_{\{x\}}} H^j(F)_x \} \quad (4.20)$$

for $x \in X$. We will show that the characteristic cycle $CC(\varphi(F, \Phi))$ of $\varphi(F, \Phi)$ (see Proposition 2.10) is equal to the Lefschetz cycle $LC(F, \Phi)$. For this purpose, we need the following.

Definition 4.6 ([17] and [20]) Let $\varphi: X \rightarrow \mathbb{Z}$ be a \mathbb{Z} -valued constructible function on X and U a relatively compact subanalytic open subset in X . We define the topological integral $\int_U \varphi$ of φ over U by

$$\int_U \varphi = \sum_{\alpha \in \mathbb{Z}} c_\alpha \cdot \chi(R\Gamma(U; \mathbb{C}_{X_\alpha})), \quad (4.21)$$

where $\varphi = \sum_{\alpha \in A} c_\alpha \mathbf{1}_{X_\alpha}$ ($c_\alpha \in \mathbb{Z}$) is an expression of φ with respect to a subanalytic stratification $X = \bigsqcup_{\alpha \in A} X_\alpha$ of X .

We can extend \mathbb{C} -linearly this integral $\int_U: \operatorname{CF}(X) \rightarrow \mathbb{Z}$ and obtain a \mathbb{C} -linear map

$$\int_U: \operatorname{CF}(X)_{\mathbb{C}} \rightarrow \mathbb{C}. \quad (4.22)$$

On the other hand, since any relatively compact subanalytic open subset U of X is invariant by $\phi = \operatorname{id}_X$, the global trace on U

$$\operatorname{tr}(F|_U, \Phi|_U) = \sum_{j \in \mathbb{Z}} (-1)^j \operatorname{tr} \{ H^j(U; F) \xrightarrow{\Phi|_U} H^j(U; F) \} \quad (4.23)$$

is well-defined.

Lemma 4.7 *For any relatively compact subanalytic open subset U of X , we have*

$$\operatorname{tr}(F|_U, \Phi|_U) = \int_U \varphi(F, \Phi). \quad (4.24)$$

The proof of this lemma being completely similar to that of [9, Proposition 11.6], we omit the proof.

Theorem 4.8 *In the situation $\phi = \text{id}_X$, $\Phi: F \longrightarrow F$ etc. as above, we have the equality*

$$LC(F, \Phi) = CC(\varphi(F, \Phi)) \quad (4.25)$$

*as Lagrangian cycles in T^*X .*

Proof. Let $X = \bigsqcup_{\alpha \in A} X_\alpha$ be a μ -stratification of X such that

$$\text{supp}(LC(F, \Phi)), \quad \text{supp}(CC(\varphi(F, \Phi))) \subset \Lambda = \bigsqcup_{\alpha \in A} T_{X_\alpha}^* X. \quad (4.26)$$

Take an open dense smooth subanalytic subset Λ_0 of Λ whose decomposition $\Lambda = \bigsqcup_{i \in I} \Lambda_i$ into connected components satisfies the condition (4.17). Let us fix Λ_i and X_{α_i} such that $\Lambda_i \subset T_{X_{\alpha_i}}^* X$. It is enough to show that $LC(F, \Phi)$ and $CC(\varphi(F, \Phi))$ coincide in an open neighborhood of Λ_i in T^*X . By Corollary 4.5, in an open neighborhood U_i of Λ_i in T^*X we have

$$LC(F, \Phi) = m_i \cdot [T_{X_{\alpha_i}}^* X], \quad (4.27)$$

where $m_i \in \mathbb{C}$ is defined by (4.18) for $p \in \Lambda_i$, $x = \pi_X(p) \in X_{\alpha_i}$, $f: X \longrightarrow \mathbb{R}$ as in Definition 4.4. Let U be a sufficiently small open ball in X centered at $x \in X_{\alpha_i}$. Set $V := U \cap \{f < f(x)\}$. Then we have

$$m_i = \sum_{j \in \mathbb{Z}} (-1)^j \text{tr}\{H_{\{f \geq f(x)\}}^j(U; F) \xrightarrow{\Phi} H_{\{f \geq f(x)\}}^j(U; F)\} \quad (4.28)$$

$$= \text{tr}(F|_U, \Phi|_U) - \text{tr}(F|_V, \Phi|_V) \quad (4.29)$$

$$= \int_U \varphi(F, \Phi) - \int_V \varphi(F, \Phi). \quad (4.30)$$

This last number coincides with the coefficient of $[T_{X_{\alpha_i}}^* X]|_{U_i}$ in $CC(\varphi(F, \Phi))|_{U_i}$. This completes the proof. \square

5 Hyperbolic localization and Lefschetz cycles

In this section, we explicitly describe the Lefschetz cycle $LC(F, \Phi)$ introduced in Section 3 in terms of hyperbolic localizations of the specializations of F . Let M be a possibly singular fixed point component of $\phi: X \longrightarrow X$. Throughout this section, we assume the conditions $\text{supp}(F) \cap M \subset M_{\text{reg}}$ and

$$“1 \notin \text{Ev}(\phi'_x) \text{ for any } x \in \text{supp}(F) \cap M_{\text{reg}}.” \quad (5.1)$$

Then there exists an open neighborhood U of $\text{supp}(F) \cap M_{\text{reg}}$ in M_{reg} such that Γ_ϕ intersects with Δ_X cleanly along $U \subset M \subset \Gamma_\phi \cap \Delta_X$. Namely, there exists a Lefschetz bundle $\mathcal{F} = U \times_M \{T_{\Gamma_\phi}^*(X \times X) \cap T_{\Delta_X}^*(X \times X)\}$ over U which is isomorphic to T^*U . As in the same way as in Section 3, we can define a Lagrangian cycle in \mathcal{F} associated with (F, Φ) . We still denote it by $LC(F, \Phi)$ and want to describe it explicitly. Replacing X, M etc. by $X \setminus (M \setminus U), U$ etc. respectively, we may assume that M is smooth and $1 \notin \text{Ev}(\phi'_x)$ for any $x \in M$ from the first. In this situation, the fixed point set of $\phi': T_M X \longrightarrow T_M X$ is the zero-section M . Let $\Gamma_{\phi'} = \{(\phi'(p), p) \mid p \in T_M X\} \subset T_M X \times T_M X$ be the graph of ϕ' and $\Delta_{T_M X} \simeq T_M X$ the diagonal subset of $T_M X \times T_M X$. Then

$$\mathcal{F}' := T_{\Gamma_{\phi'}}^*(T_M X \times T_M X) \cap T_{\Delta_{T_M X}}^*(T_M X \times T_M X) \quad (5.2)$$

is a vector bundle over the zero-section $M \simeq \Gamma_{\phi'} \cap \Delta_{T_M X}$ of $T_M X$. Since \mathcal{F}' is also isomorphic to T^*M by our assumptions, we shall identify it with the original Lefschetz bundle $\mathcal{F} = T_{\Gamma_\phi}^*(X \times X) \cap T_{\Delta_X}^*(X \times X)$. Now consider the natural morphism

$$\Phi': (\phi')^{-1}\nu_M(F) \longrightarrow \nu_M(F) \quad (5.3)$$

induced by $\Phi: \phi^{-1}F \longrightarrow F$. Then from the pair $(\nu_M(F), \Phi')$, we can construct the Lefschetz cycle $LC(\nu_M(F), \Phi')$ in $\mathcal{F}' \simeq \mathcal{F}$.

Proposition 5.1 *In $\mathcal{F} \simeq \mathcal{F}'$, we have*

$$LC(F, \Phi) = LC(\nu_M(F), \Phi'). \quad (5.4)$$

Proof. First, we briefly recall the proof of Proposition 2.13 which is similar to that of [17, Proposition 9.6.11]. Since the construction of the characteristic class $C(F, \Phi)_M \in H_{\text{supp}(F) \cap M}^0(X; \omega_X)$ is local around $\text{supp}(F) \cap M$ (see [17, Remark 9.6.7]) and $X \setminus (M \setminus U)$ is invariant by ϕ , we may replace X, M etc. by $X \setminus (M \setminus U), U$ etc. respectively. Then the proof follows from the commutativity of the diagram (5.7) below. Here we denote $T_M X$ simply by \mathcal{G} and the morphism $\tilde{h}: T_M X \longrightarrow T_M X \times T_M X$ is defined by $\tilde{h} = (\phi', \text{id})$. We also used the natural isomorphism $D\nu_M(F) \simeq \nu_M(DF)$. Let us explain the construction of the morphism **A** in the diagram (5.7). Consider the commutative diagram:

$$\begin{array}{ccccccc} T_{M \times M}(X \times X) & \xrightarrow{s_1} & \widetilde{(X \times X)}_{M \times M} & \xleftarrow{j_1} & \Omega_{X \times X} & \xrightarrow{\tilde{p}_1} & X \times X \\ \delta_{T_M X} \uparrow & & \delta' \uparrow & & \delta \uparrow & & \delta_X \uparrow \\ & \square & & \square & & \square & \\ T_M X & \xrightarrow{s} & \widetilde{X}_M & \xleftarrow{j} & \Omega_X & \xrightarrow{\tilde{p}} & X, \end{array} \quad (5.5)$$

where $\widetilde{(X \times X)}_{M \times M}$ is the normal deformation of $X \times X$ along $M \times M$ and $t_1: \widetilde{(X \times X)}_{M \times M} \longrightarrow \mathbb{R}$ is the deformation parameter such that $\Omega_{X \times X}$ is defined by $t_1 > 0$ in $(X \times X)_{M \times M}$. Then the morphism **A** is constructed by the morphisms of functors

$$\begin{aligned} \delta_X^! &\longrightarrow \delta_X^! R\tilde{p}_{1*} \tilde{p}_1^{-1} \simeq R\tilde{p}_* \tilde{\delta}^! \tilde{p}_1^{-1} \simeq R p_* \tilde{\delta}'^! R j_{1*} \tilde{p}_1^{-1} \\ &\longrightarrow R p_* \tilde{\delta}'^! s_{1*} s_1^{-1} R j_{1*} \tilde{p}_1^{-1} \simeq R p_* s_* \delta_{T_M X}^! s_1^{-1} R j_{1*} \tilde{p}_1^{-1}. \end{aligned} \quad (5.6)$$

The other horizontal arrows in the diagram (5.7) are constructed similarly.

$$\begin{array}{ccccc}
R\mathrm{Hom}(F, F) & \xrightarrow{\hspace{10em}} & R\mathrm{Hom}(\nu_M(F), \nu_M(F)) & & (5.7) \\
\uparrow \wr & & \uparrow \wr & & \\
R\Gamma_{\Delta_X}(X \times X; F \boxtimes DF) & \xrightarrow{\mathbf{A}} & R\Gamma_{\Delta_{\mathcal{G}}}(\mathcal{G} \times \mathcal{G}; \nu_{M \times M}(F \boxtimes DF)) & \longleftarrow & R\Gamma_{\Delta_{\mathcal{G}}}(\mathcal{G} \times \mathcal{G}; \nu_M(F) \boxtimes D\nu_M(F)) \\
\downarrow & & \downarrow & & \downarrow \\
& & R\Gamma_M(\mathcal{G}; \tilde{h}^{-1} \nu_{M \times M}(F \boxtimes DF)) & \longleftarrow & R\Gamma_M(\mathcal{G}; \phi'^{-1} \nu_M(F) \otimes D\nu_M(F)) \\
\downarrow & & \downarrow & & \downarrow \\
R\Gamma_M(X; \phi^{-1} F \otimes DF) & \xrightarrow{\sim} & R\Gamma_M(\mathcal{G}; \nu_M(\phi^{-1} F \otimes DF)) & \longleftarrow & R\Gamma_M(\mathcal{G}; \nu_M(\phi^{-1} F) \otimes D\nu_M(F)) \\
\downarrow \Phi & & \downarrow \Phi & & \downarrow \Phi \\
R\Gamma_M(X; F \otimes DF) & \xrightarrow{\sim} & R\Gamma_M(\mathcal{G}; \nu_M(F \otimes DF)) & \longleftarrow & R\Gamma_M(\mathcal{G}; \nu_M(F) \otimes D\nu_M(F)) \\
\downarrow & & \downarrow & & \downarrow \\
R\Gamma_M(X; \omega_X) & \xrightarrow{\sim} & R\Gamma_M(\mathcal{G}; \nu_M(\omega_X)) & \xrightarrow{\sim} & R\Gamma_M(\mathcal{G}; \omega_{\mathcal{G}}) \\
\downarrow & & & & \downarrow \\
\mathbb{C} & \xlongequal{\hspace{10em}} & & & \mathbb{C}.
\end{array}$$

Now the proof of Proposition 5.1 follows from the commutativity of Diagram 5.1.a below, which is a microlocal version of Diagram (5.7). Here we denote $T_M X$, $\mathrm{SS}(F)$ and $C_{T_M^* X}(\mathrm{SS}(F))$ by \mathcal{G} , S and S' respectively. Note that we have natural isomorphisms

$$T^*(T_M X) \simeq T^*(T_M^* X) \simeq T_{T_M^* X}^*(T^* X) \quad (5.8)$$

(see [17, (6.2.3)] and (5.22) below) and the normal cone $S' = C_{T_M^* X}(\mathrm{SS}(F))$ can be considered as a subset of $T^*(T_M X) = T^* \mathcal{G}$. We also used a conic isotropic subset $S'' = (S \cap \mathcal{F}) \cup (S' \cap \mathcal{F}')$ of $\mathcal{F} \simeq \mathcal{F}' \simeq T^* M$ and the morphism $\tilde{h}: T_M X \rightarrow T_M X \times T_M X$ is defined by $\tilde{h} = (\phi', \mathrm{id}_{T_M X})$. Moreover we used the natural isomorphism $D\nu_M(F) \simeq \nu_M(DF)$ to obtain Diagram 5.1.a. Let us explain the construction of the morphism \mathbf{A} in Diagram 5.1.a. First consider the commutative diagram:

$$\begin{array}{ccccccc}
T_{M \times M}(X \times X) & \xhookrightarrow{s_1} & \widetilde{(X \times X)}_{M \times M} & \xleftarrow{j_1} & \Omega_{X \times X} & \xrightarrow{\tilde{p}_1} & X \times X \\
\delta_{T_M X} \uparrow & \square & \tilde{\delta}' \uparrow & \square & \tilde{\delta} \uparrow & \square & \delta_X \uparrow \\
T_M X & \xhookrightarrow{s} & \widetilde{X}_M & \xleftarrow{j} & \Omega_X & \xrightarrow{\tilde{p}} & X
\end{array} \quad (5.9)$$

which already appeared in the proof of Proposition 2.13. Denote the image of $\tilde{\delta}'$ (resp. $\tilde{\delta}$) by $\Delta_{\widetilde{X}_M}$ (resp. Δ_{Ω_X}). Then we see that the following morphisms are isomorphisms.

$${}^t \tilde{p}_1' : \Delta_{\Omega_X} \times_{\Delta_X} T_{\Delta_X}^*(X \times X) \rightarrow T_{\Delta_{\Omega_X}}^* \Omega_{X \times X}, \quad (5.10)$$

$${}^t j_1' : \Delta_{\Omega_X} \times_{\Delta_{\widetilde{X}_M}} T_{\Delta_{\widetilde{X}_M}}^*((X \times X)_{M \times M}) \rightarrow T_{\Delta_{\Omega_X}}^* \Omega_{X \times X}, \quad (5.11)$$

$${}^t s_1' : \Delta_{T_M X} \times_{\Delta_{\widetilde{X}_M}} T_{\Delta_{\widetilde{X}_M}}^*((X \times X)_{M \times M}) \rightarrow T_{\Delta_{T_M X}}^*(T_{M \times M}(X \times X)). \quad (5.12)$$

$$\begin{array}{ccccc}
R\mathrm{Hom}(F, F) & \xrightarrow{\hspace{15em}} & R\mathrm{Hom}(\nu_M(F), \nu_M(F)) \\
\wr \downarrow & & \wr \downarrow \\
R\Gamma_S(T^*X; \mu_{\Delta_X}(F \boxtimes DF)) & \xrightarrow{\mathbf{A}} R\Gamma_{S'}(T^*\mathcal{G}; \mu_{\Delta_{\mathcal{G}}}(\nu_{M \times M}(F \boxtimes DF))) & \xleftarrow{\hspace{1em}} & R\Gamma_{S'}(T^*\mathcal{G}; \mu_{\Delta_{\mathcal{G}}}(\nu_M(F) \boxtimes D\nu_M(F))) \\
\downarrow & \downarrow & & \downarrow \\
& R\Gamma_{S'}(T^*\mathcal{G}; \mu_{\Delta_{\mathcal{G}}}(\tilde{h}_* \tilde{h}^{-1} \nu_{M \times M}(F \boxtimes DF))) & \xleftarrow{\hspace{1em}} & R\Gamma_{S'}(T^*\mathcal{G}; \mu_{\Delta_{\mathcal{G}}}(\tilde{h}_*(\phi'^{-1} \nu_M(F) \otimes D\nu_M(F)))) \\
& \downarrow & & \downarrow \\
R\Gamma_S(T^*X; \mu_{\Delta_X}(h_*(\phi^{-1}F \otimes DF))) & \xrightarrow{\mathbf{B}} R\Gamma_{S'}(T^*\mathcal{G}; \mu_{\Delta_{\mathcal{G}}}(\tilde{h}_* \nu_M(\phi^{-1}F \otimes DF))) & \xleftarrow{\hspace{1em}} & R\Gamma_{S'}(T^*\mathcal{G}; \mu_{\Delta_{\mathcal{G}}}(\tilde{h}_*(\nu_M(\phi^{-1}F) \otimes D\nu_M(F)))) \\
\downarrow \Phi & \downarrow \Phi & & \downarrow \Phi \\
R\Gamma_S(T^*X; \mu_{\Delta_X}(h_*(F \otimes DF))) & \longrightarrow R\Gamma_{S'}(T^*\mathcal{G}; \mu_{\Delta_{\mathcal{G}}}(\tilde{h}_* \nu_M(F \otimes DF))) & \xleftarrow{\hspace{1em}} & R\Gamma_{S'}(T^*\mathcal{G}; \mu_{\Delta_{\mathcal{G}}}(\tilde{h}_*(\nu_M(F) \otimes D\nu_M(F)))) \\
\downarrow & \downarrow & & \downarrow \\
R\Gamma_S(T^*X; \mu_{\Delta_X}(h_*\omega_X)) & \longrightarrow R\Gamma_{S'}(T^*\mathcal{G}; \mu_{\Delta_{\mathcal{G}}}(\tilde{h}_* \nu_M(\omega_X))) & \xrightarrow{\sim} & R\Gamma_{S'}(T^*\mathcal{G}; \mu_{\Delta_{\mathcal{G}}}(\tilde{h}_*\omega_{\mathcal{G}})) \\
\downarrow & \downarrow & & \downarrow \\
R\Gamma_{S''}(\mathcal{F}; \pi_M^{-1}\omega_M) & \xlongequal{\hspace{1.5cm}} R\Gamma_{S''}(\mathcal{F}; \pi_M^{-1}\omega_M) & \xlongequal{\hspace{1.5cm}} & R\Gamma_{S''}(\mathcal{F}; \pi_M^{-1}\omega_M).
\end{array}$$

Now let us set

$$S_1 := {}^t\tilde{p}_1'(\Delta_{\Omega_X} \times_{\Delta_X} S), \quad (5.13)$$

$$S_2 := \overline{{}^tj_1'^{-1}(S_1)}, \quad (5.14)$$

$$S_3 := S_2 \cap T_{\Delta_{T_M X}}^*(T_{M \times M}(X \times X)). \quad (5.15)$$

Then we have the following morphisms

$$\begin{aligned} R\Gamma_{\text{SS}(F)}(T_{\Delta_X}^*(X \times X); \mu_{\Delta_X}(F \boxtimes DF)) \\ \longrightarrow R\Gamma_{\text{SS}(F)}(T_{\Delta_X}^*(X \times X); \mu_{\Delta_X}(R\tilde{p}_{1*}\tilde{p}_1^{-1}(F \boxtimes DF))) \end{aligned} \quad (5.16)$$

$$\longrightarrow R\Gamma_{S_1}(T_{\Delta_{\Omega_X}}^*\Omega_{X \times X}; \mu_{\Delta_{\Omega_X}}(\tilde{p}_1^{-1}(F \boxtimes DF))) \quad (5.17)$$

$$\xleftarrow{\sim} R\Gamma_{S_2}(T_{\Delta_{\widehat{X_M}}}^*((X \times X)_{M \times M}); \mu_{\Delta_{\widehat{X_M}}}(\tilde{p}_1^{-1}(F \boxtimes DF))) \quad (5.18)$$

$$\longrightarrow R\Gamma_{S_2}(T_{\Delta_{\widehat{X_M}}}^*((X \times X)_{M \times M}); \mu_{\Delta_{\widehat{X_M}}}(s_{1*}s_1^{-1}Rj_{1*}\tilde{p}_1^{-1}(F \boxtimes DF))) \quad (5.19)$$

$$\longrightarrow R\Gamma_{S_3}(T_{\Delta_{T_M X}}^*(T_{M \times M}(X \times X)); \mu_{\Delta_{T_M X}}(s_1^{-1}Rj_{1*}\tilde{p}_1^{-1}(F \boxtimes DF))) \quad (5.20)$$

$$= R\Gamma_{S_3}(T^*\mathcal{G}; \mu_{\Delta_{\mathcal{G}}}(\nu_{M \times M}(F \boxtimes DF))), \quad (5.21)$$

where we used [17, Theorem 4.3.2 and Proposition 3.3.9] (see also the arguments in [17, page 192-193]) to prove that the morphism (5.18) is an isomorphism. Let us show that S_3 is equal to S' . Let (x', x'') be a local coordinate system of X such that $M = \{x' = 0\}$ and $(x', x''; \xi', \xi'')$ the associated coordinates of T^*X . Then by the Hamiltonian isomorphism etc., we can naturally identify $T^*(T_M X) \simeq T_{\Delta_{T_M X}}^*(T_{M \times M}(X \times X))$ with $T_{T_M X}^*(T^*X)$ as follows (see [17, (6.2.3)]).

$$\begin{array}{ccccc} T^*(T_M X) & \simeq & T^*(T_M^* X) & \simeq & T_{T_M^* X}^*(T^* X). \\ \downarrow \Psi & & \downarrow \Psi & & \downarrow \Psi \\ (x', x''; \xi', \xi'') & \longleftrightarrow & (\xi', x''; -x', \xi'') & \longleftrightarrow & (x', x''; \xi', \xi'') \end{array} \quad (5.22)$$

Under this identification, we can prove that $S_3 \subset T^*(T_M X) \simeq T_{\Delta_{T_M X}}^*(T_{M \times M}(X \times X))$ is equal to the normal cone $S' = C_{T_M^* X}(\text{SS}(F)) \subset T_{T_M^* X}^*(T^* X)$ as follows. In the associated local coordinates $(x', x'', t; \xi', \xi'')$ ($t > 0$) of $\Delta_{\Omega_X} \times_{\Delta_{\widehat{X_M}}} T_{\Delta_{\widehat{X_M}}}^*((X \times X)_{M \times M}) (\simeq \Delta_{\Omega_X} \times_{\Delta_X} T_{\Delta_X}^*(X \times X) \simeq \Omega_X \times_X T^*X)$, its subset ${}^tj_1'^{-1}t\tilde{p}_1'(\Delta_{\Omega_X} \times_{\Delta_X} S)$ is expressed by

$$\{(x', x'', t; \xi', \xi'') \in \Delta_{\Omega_X} \times_{\Delta_{\widehat{X_M}}} T_{\Delta_{\widehat{X_M}}}^*((X \times X)_{M \times M}) \mid (tx', x''; t^{-1}\xi', \xi'') \in \text{SS}(F)\}. \quad (5.23)$$

Hence we have

$$\begin{aligned} (x', x''; \xi', \xi'') \in S_3 = S_2 \cap T_{\Delta_{T_M X}}^*(T_{M \times M}(X \times X)) \\ \iff \exists (x'_n, x''_n, t_n; \xi'_n, \xi''_n) \in \Delta_{\Omega_X} \times_{\Delta_{\widehat{X_M}}} T_{\Delta_{\widehat{X_M}}}^*((X \times X)_{M \times M}) \\ \text{s.t. } \begin{cases} (x'_n, x''_n, t_n; \xi'_n, \xi''_n) \xrightarrow{n \rightarrow \infty} (x', x'', 0; \xi', \xi''), \\ (t_n x'_n, x''_n; t_n^{-1} \xi'_n, \xi''_n) \in \text{SS}(F) \end{cases} \end{aligned} \quad (5.24)$$

$$\begin{aligned} \iff \exists (x'_n, x''_n, t_n; \xi'_n, \xi''_n) \in \Delta_{\Omega_X} \times_{\Delta_{\widehat{X_M}}} T_{\Delta_{\widehat{X_M}}}^*((X \times X)_{M \times M}) \\ \text{s.t. } \begin{cases} (x'_n, x''_n, t_n; \xi'_n, \xi''_n) \xrightarrow{n \rightarrow \infty} (x', x'', 0; \xi', \xi''), \\ (t_n x'_n, x''_n; \xi'_n, t_n \xi''_n) \in \text{SS}(F) \end{cases} \end{aligned} \quad (5.25)$$

$$\begin{aligned} \iff \exists ((\widetilde{x}_n', \widetilde{x}_n''; \widetilde{\xi}_n', \widetilde{\xi}_n''), c_n) \in \text{SS}(F) \times \mathbb{R}_{>0} \\ \text{s.t. } \begin{cases} (\widetilde{x}_n', \widetilde{x}_n''; \widetilde{\xi}_n', \widetilde{\xi}_n'') \xrightarrow{n \rightarrow \infty} (0, x''; \xi', 0), \\ (c_n \widetilde{x}_n', c_n \widetilde{\xi}_n'') \xrightarrow{n \rightarrow \infty} (x', \xi'') \end{cases} \end{aligned} \quad (5.26)$$

$$\iff (x', x''; \xi', \xi'') \in S' = C_{T_M^* X}(\text{SS}(F)) \subset T_{T_M^* X} T^* X. \quad (5.27)$$

We thus obtained the morphism **A**:

$$R\Gamma_S(T^* X; \mu_{\Delta_X}(F \boxtimes DF)) \longrightarrow R\Gamma_{S'}(T^* \mathcal{G}; \mu_{\Delta_{\mathcal{G}}}(\nu_{M \times M}(F \boxtimes DF))). \quad (5.28)$$

We can construct also the morphism **B** in Diagram 5.1.a as follows.

$$\begin{aligned} R\Gamma_S(T^* X; \mu_{\Delta_X}(h_*(\phi^{-1} F \otimes DF))) \\ \longrightarrow R\Gamma_{S'}(T^* \mathcal{G}; \mu_{\Delta_{\mathcal{G}}}(\nu_{M \times M}(h_*(\phi^{-1} F \otimes DF)))) \end{aligned} \quad (5.29)$$

$$\longrightarrow R\Gamma_{S'}(T^* \mathcal{G}; \mu_{\Delta_{\mathcal{G}}}(\widetilde{h}_* \nu_M(\phi^{-1} F \otimes DF))), \quad (5.30)$$

where the first morphism is constructed in the same way as **A** and we used [17, Proposition 4.2.4] to construct the second morphism. This completes the proof. \square

In what follows, we shall identify $\mathcal{F} \simeq \mathcal{F}'$ with $T^* M$ and describe $LC(F, \Phi) = LC(\nu_M(F), \Phi')$. Since our result holds for any conic object on any vector bundle over M , let us consider the following general setting. Let $\tau: \mathcal{G} \rightarrow M$ be a real vector bundle of rank $r > 0$ over M and $\psi: \mathcal{G} \rightarrow \mathcal{G}$ its endomorphism. Assume that the fixed point set of ψ is the zero-section M of \mathcal{G} . This assumption implies that

$$1 \notin \text{Ev}(\psi_x) \text{ for any } x \in M. \quad (5.31)$$

Suppose that we are given a conic \mathbb{R} -constructible object $G \in \mathbf{D}_{\mathbb{R}-c}^b(\mathcal{G})$ on \mathcal{G} and a morphism $\Psi: \psi^{-1} G \rightarrow G$ in $\mathbf{D}_{\mathbb{R}-c}^b(\mathcal{G})$. From these data, we can construct the Lefschetz bundle $\mathcal{F}_0 \simeq T^* M$ associated with ψ and the Lefschetz cycle $LC(G, \Psi)$ in it.

Fix a point $\overset{\circ}{x} \in M$ and consider the linear homomorphism $\psi_{\overset{\circ}{x}}: \mathcal{G}_{\overset{\circ}{x}} \rightarrow \mathcal{G}_{\overset{\circ}{x}}$. Let $\lambda_1, \dots, \lambda_d$ be the eigenvalues of $\psi_{\overset{\circ}{x}}$ on $\mathbb{R}_{\geq 1}$ and $\lambda_{d+1}, \dots, \lambda_r$ the remaining ones. Since these eigenvalues vary depending on $x \in M$ continuously, we denote their continuous extensions to a neighborhood of $\overset{\circ}{x}$ in M by $\lambda_1(x), \dots, \lambda_r(x)$. Then for a sufficiently large $R > 0$ we have $\lambda_1, \dots, \lambda_d \in [1, R]$. Moreover there exists a sufficiently small $\varepsilon > 0$ such that

$$\lambda_{d+1}, \dots, \lambda_r \notin \{z \in \mathbb{C} \mid \text{Re } z \geq 1, |\text{Im } z| \leq \varepsilon\}. \quad (5.32)$$

By the continuity of the eigenvalues, there exists a sufficiently small neighborhood U of $\overset{\circ}{x}$ in M such that

$$\lambda_{d+1}(x), \dots, \lambda_r(x) \notin \{z \in \mathbb{C} \mid \text{Re } z \geq 1, |\text{Im } z| \leq \varepsilon\}, \quad (5.33)$$

$$\lambda_1(x), \dots, \lambda_d(x) \in \{z \in \mathbb{C} \mid 1 < \text{Re } z < R + 1, |\text{Im } z| < \varepsilon\} \quad (5.34)$$

for any $x \in U$. If necessary, replacing U by a smaller one, we may assume also that \mathcal{G} is trivial on U . For $x \in U$ we set

$$P_x = \frac{1}{2\pi i} \int_C (z - \psi_x)^{-1} dz, \quad (5.35)$$

where C is the path on the boundary of the set $\{z \in \mathbb{C} \mid 1 < \operatorname{Re} z < R+1, |\operatorname{Im} z| < \varepsilon\} \subset \mathbb{C}$. Then $P_x: \mathcal{G}_x \rightarrow \mathcal{G}_x$ is the projector onto the direct sum of the generalized eigenspaces associated with the eigenvalues in $\{z \in \mathbb{C} \mid 1 < \operatorname{Re} z < R+1, |\operatorname{Im} z| < \varepsilon\} \subset \mathbb{C}$. The family $\{P_x\}_{x \in U}$ defines an endomorphism P of $\mathcal{G}|_U$, whose image $\mathcal{W} \subset \mathcal{G}|_U$ is a subbundle of $\mathcal{G}|_U$.

Definition 5.2 We call $\mathcal{W} := \operatorname{Im} P \subset \mathcal{G}|_U$ the minimal expanding subbundle of $\mathcal{G}|_U$ (on the neighborhood U of $\overset{\circ}{x} \in M$).

Definition 5.3 ([17, Section 9.6]) We say that a subbundle \mathcal{E} of $\mathcal{G}|_U$ is an expanding subbundle if it satisfies the following conditions:

- (i) $\psi|_{\tau^{-1}(U)}(\mathcal{E}) \subset \mathcal{E}$.
- (ii) \mathcal{W} is a subbundle of \mathcal{E} .
- (iii) $\mathcal{E}_x^{\mathbb{C}} \subset \bigoplus_{\lambda \notin [0,1]} (\mathcal{G}_x)_{\lambda}^{\mathbb{C}}$ for any $x \in U$.

For any expanding subbundle \mathcal{E} of $\mathcal{G}|_U$, the induced morphism $\psi|_{\mathcal{E}}: \mathcal{E} \rightarrow \mathcal{E}$ is an isomorphism of vector bundles.

Definition 5.4 ([1]) Let $\tau_{\mathcal{E}}: \mathcal{E} \rightarrow U$ be an expanding subbundle of $\mathcal{G}|_U$ and $i_{\mathcal{E}}: U \hookrightarrow \mathcal{E}$ its zero-section. We define an object $G_{\mathcal{E}}^{\dagger-1} \in \mathbf{D}_{\mathbb{R}-c}^b(U)$ by

$$G_{\mathcal{E}}^{\dagger-1} := i_{\mathcal{E}}^{\dagger}(G|_{\mathcal{E}}) \simeq R\tau_{\mathcal{E}!}(G|_{\mathcal{E}}) \quad (5.36)$$

and its endomorphism $\Psi_{\mathcal{E}}^{\dagger-1}: G_{\mathcal{E}}^{\dagger-1} \rightarrow G_{\mathcal{E}}^{\dagger-1}$ by the composite of the morphisms

$$R\tau_{\mathcal{E}!}(G|_{\mathcal{E}}) \rightarrow R\tau_{\mathcal{E}!}\tilde{\psi}_*\tilde{\psi}^{-1}(G|_{\mathcal{E}}) \quad (5.37)$$

$$\simeq R\tau_{\mathcal{E}!}\tilde{\psi}_!((\psi^{-1}G)|_{\mathcal{E}}) \simeq R\tau_{\mathcal{E}!}((\psi^{-1}G)|_{\mathcal{E}}) \quad (5.38)$$

$$\xrightarrow{\Psi} R\tau_{\mathcal{E}!}(G|_{\mathcal{E}}). \quad (5.39)$$

Here we set $\tilde{\psi} := \psi|_{\mathcal{E}}$ and the first morphism above is induced by the adjunction. We call the pair $(G_{\mathcal{E}}^{\dagger-1}, \Psi_{\mathcal{E}}^{\dagger-1})$ the hyperbolic localization of (G, Ψ) with respect to \mathcal{E} .

Proposition 5.5 Let $\overset{\circ}{x} \in M$ be a point of M . Then there exists a sufficiently small open neighborhood U of $\overset{\circ}{x}$ in M such that for any compact subanalytic subset K of U and for any expanding subbundle \mathcal{E} of $\mathcal{G}|_U$, we have

$$\int_{\tilde{U}} C(G_{\tilde{K}}|_{\tilde{U}}, \Psi_{\tilde{K}}|_{\tilde{U}}) = \operatorname{tr}((G_{\tilde{K}})_{\mathcal{E}}^{\dagger-1}, (\Psi_{\tilde{K}})_{\mathcal{E}}^{\dagger-1}). \quad (5.40)$$

Here we set $\tilde{U} := \tau^{-1}(U)$ and $\tilde{K} := \tau^{-1}(K)$.

The proof of this proposition is completely similar to that of [17, Proposition 9.6.12] and we omit it here. By the isomorphism $(G_{\tilde{K}})_{\mathcal{E}}^{\dagger-1} \simeq (G_{\mathcal{E}}^{\dagger-1})_K$, we thus obtain the equality

$$\int_{\tilde{U}} C(G_{\tilde{K}}|_{\tilde{U}}, \Psi_{\tilde{K}}|_{\tilde{U}}) = \operatorname{tr}((G_{\mathcal{E}}^{\dagger-1})_K, (\Psi_{\mathcal{E}}^{\dagger-1})_K). \quad (5.41)$$

Take a sufficiently small open subset U of M for which Proposition 5.5 holds and define a constructible function $\varphi(G_{\mathcal{E}}^{l-1}, \Psi_{\mathcal{E}}^{l-1})$ on it associated to the hyperbolic localization $(G_{\mathcal{E}}^{l-1}, \Psi_{\mathcal{E}}^{l-1})$ by

$$\varphi(G_{\mathcal{E}}^{l-1}, \Psi_{\mathcal{E}}^{l-1})(x) := \sum_{j \in \mathbb{Z}} (-1)^j \text{tr} \{ H^j(G_{\mathcal{E}}^{l-1})_x \xrightarrow{\Psi_{\mathcal{E}}^{l-1}|_{\{x\}}} H^j(G_{\mathcal{E}}^{l-1})_x \}. \quad (5.42)$$

Then by applying (5.41) to the special case where K is a point, we find that it does not depend on the choice of the expanding subbundle \mathcal{E} . Hence we can glue such locally defined constructible functions to obtain a global one $\varphi_M(G, \Psi)$ on M .

Proposition 5.6 *Under the condition (5.31) we have the equality*

$$LC(G, \Psi) = CC(\varphi_M(G, \Psi)) \quad (5.43)$$

as Lagrangian cycles in T^*M .

Proof. Let $\pi_M: T^*M \rightarrow M$ be the projection. Fix a point $\overset{\circ}{x} \in M$ and compare the both sides of (5.43) on a neighborhood of $\pi_M^{-1}(\overset{\circ}{x}) \subset T^*M$. By the homotopy invariance of Lefschetz cycles (see Proposition 3.8), [17, Proposition 9.6.8] and (5.41), taking a sufficiently small open neighborhood U of $\overset{\circ}{x}$ and replacing ψ with $t\psi$ for $|1-t| \ll 1$, we may assume the following conditions:

- (1) $\mathcal{G}|_U$ is trivial.
- (2) The open subset U satisfies the condition of Proposition 5.5.
- (3) $\text{Ev}(\psi_x) \cap \{z \in \mathbb{C} \mid |z| = 1\} = \emptyset$ for any $x \in U$.

It is enough to show that

$$LC(G|_{\tilde{U}}, \Psi|_{\tilde{U}}) = CC(\varphi_M(G, \Psi)|_U), \quad (5.44)$$

where $\tilde{U} = \tau^{-1}(U)$. As in the proof of [17, Proposition 9.6.12] we can construct subbundles \mathcal{G}_+ and \mathcal{G}_- of $\mathcal{G}|_U$ for which we have the direct sum decomposition $\mathcal{G}|_U = \mathcal{G}_+ \oplus \mathcal{G}_-$ and a metric on $\mathcal{G}|_U$ such that

$$\begin{aligned} &\text{there exist constants } c_1, c_2 \text{ with } 0 < c_1 < 1 < c_2 \text{ satisfying the condition} \\ &|\psi_x(v_-)| \leq c_1 |v_-| \ (v_- \in \mathcal{G}_{-,x}), \ |\psi_x(v_+)| \geq c_2 |v_+| \ (v_+ \in \mathcal{G}_{+,x}) \text{ for any } x \in U. \end{aligned} \quad (5.45)$$

By using this metric, we set

$$Z := \{(x, v_+, v_-) \in \mathcal{G}|_U \mid |v_+| < a, |v_-| \leq b\}, \quad (5.46)$$

for some fixed constants $a, b > 0$. Then $\psi^{-1}(Z) \cap Z$ is open in Z and closed in $\psi^{-1}(Z)$ and hence we can construct a morphism

$$\Psi_Z: \psi^{-1}(G_Z) \rightarrow G_Z \quad (5.47)$$

induced by $\Psi: \psi^{-1}G \rightarrow G$. Since \mathcal{G}_+ is an expanding subbundle of $\mathcal{G}|_U$, we have

$$\varphi_M(G, \Psi)|_U = \varphi(G_{\mathcal{G}_+}^{l-1}, \Psi_{\mathcal{G}_+}^{l-1}). \quad (5.48)$$

Moreover we have

$$CC(\varphi(G_{\mathcal{G}_+}^{l-1}, \Psi_{\mathcal{G}_+}^{l-1})) = LC(G_{\mathcal{G}_+}^{l-1}, \Psi_{\mathcal{G}_+}^{l-1}), \quad (5.49)$$

by Proposition 4.8. Thus we have to show that

$$LC(G|_{\tilde{U}}, \Psi|_{\tilde{U}}) = LC(G_{\mathcal{G}_+}^{l-1}, \Psi_{\mathcal{G}_+}^{l-1}) \quad (5.50)$$

as Lagrangian cycles in T^*U . In what follows, for simplicity we write M instead of U and $\mathcal{G}|_U = \tilde{U}, G|_{\tilde{U}}, \Psi|_{\tilde{U}}$ etc. by \mathcal{G}, G, Ψ etc. respectively.

Let us take a μ -stratification $\mathcal{G} = \bigsqcup_{\alpha \in A} \mathcal{G}_\alpha$ of \mathcal{G} which satisfies the following three conditions.

- (i) There exists a subset $B \subset A$ such that the zero-section $M \subset \mathcal{G}$ of \mathcal{G} is $\bigsqcup_{\beta \in B} \mathcal{G}_\beta$.
- (ii) $\text{SS}(G_Z) \subset \bigsqcup_{\alpha \in A} T_{\mathcal{G}_\alpha}^* \mathcal{G}$ in $T^* \mathcal{G}$.
- (iii) $\text{SS}(G_{\mathcal{G}_+}^{l-1}), \text{SS}(R\tau_* G_Z) \subset \bigsqcup_{\beta \in B} T_{\mathcal{G}_\beta}^* M$ in $T^* M$.

For $\beta \in B$, we shall denote $\mathcal{G}_\beta \subset M$ by M_β . Namely $M = \bigsqcup_{\beta \in B} M_\beta$ is a μ -stratification of M . Set $\Lambda = \bigsqcup_{\beta \in B} T_{M_\beta}^* M \subset T^* M$. By the conditions above, we obtain

$$\text{supp}(LC(G, \Psi)), \quad \text{supp}(LC(G_{\mathcal{G}_+}^{l-1}, \Psi_{\mathcal{G}_+}^{l-1})) \subset \Lambda. \quad (5.51)$$

Therefore it suffices to show that $LC(G, \Psi)$ coincides with $LC(G_{\mathcal{G}_+}^{l-1}, \Psi_{\mathcal{G}_+}^{l-1})$ on an open dense subset of Λ . Let Λ_0 be an open dense smooth subanalytic subset of Λ whose decomposition $\Lambda_0 = \bigsqcup_{i \in I} \Lambda_i$ into connected components satisfies the condition

$$\text{“For any } i \in I, \text{ there exists } \beta_i \in B \text{ such that } \Lambda_i \subset T_{M_{\beta_i}}^* M.” \quad (5.52)$$

Let us fix Λ_i and M_{β_i} as above and compare $LC(G, \Psi)$ with $LC(G_{\mathcal{G}_+}^{l-1}, \Psi_{\mathcal{G}_+}^{l-1})$ on Λ_i . Take a point $p_0 \in \Lambda_i$ and set $x_0 = \pi_M(p_0) \in M_{\beta_i}$. Let $f: M \rightarrow \mathbb{R}$ be a real analytic function (defined in an open neighborhood of x_0) which satisfies that $p_0 = (x_0; df(x_0)) \in \Lambda_i$, $f(x_0) = 0$ and the Hessian $\text{Hess}(f|_{M_{\beta_i}})$ is positive definite. Then by Corollary 4.5, we have

$$LC(G_{\mathcal{G}_+}^{l-1}, \Psi_{\mathcal{G}_+}^{l-1}) = m_i \cdot [T_{M_{\beta_i}}^* M] \quad (5.53)$$

in an open neighborhood of Λ_i in $T^* M$, where $m_i \in \mathbb{C}$ is defined by

$$m_i := \sum_{j \in \mathbb{Z}} (-1)^j \text{tr} \{ H_{\{f \geq 0\}}^j(B(x_0, \delta); G_{\mathcal{G}_+}^{l-1}) \xrightarrow{\Psi_{\mathcal{G}_+}^{l-1}} H_{\{f \geq 0\}}^j(B(x_0, \delta); G_{\mathcal{G}_+}^{l-1}) \} \quad (5.54)$$

for sufficiently small $\delta > 0$. Set $V := B(x_0, \delta)$ and $W := V \cap \{f < 0\}$ in M . Then we have

$$m_i = \text{tr}(R\Gamma_V(G_{\mathcal{G}_+}^{l-1}), R\Gamma_V(\Psi_{\mathcal{G}_+}^{l-1})) - \text{tr}(R\Gamma_W(G_{\mathcal{G}_+}^{l-1}), R\Gamma_W(\Psi_{\mathcal{G}_+}^{l-1})). \quad (5.55)$$

Set also $\tilde{V} := \tau^{-1}(V)$, $\tilde{W} := \tau^{-1}(W) \subset \mathcal{G}$ and $\tilde{f} := f \circ \tau: \mathcal{G} \rightarrow \mathbb{R}$. Since we work in a sufficiently small open neighborhood of x_0 , we may assume $M = \mathbb{R}^m, x_0 = 0$. Set $g(x) := |x|^2 = x_1^2 + \cdots + x_m^2$.

Lemma 5.7 *For a sufficiently small $\delta > 0$, we have*

$$\mathrm{tr}(R\Gamma_V(G_{\mathcal{G}_+}^{!-1}), R\Gamma_V(\Psi_{\mathcal{G}_+}^{!-1})) = \mathrm{tr}(R\Gamma_{\tilde{V}}(G_Z), R\Gamma_{\tilde{V}}(\Psi_Z)), \quad (5.56)$$

$$\mathrm{tr}(R\Gamma_W(G_{\mathcal{G}_+}^{!-1}), R\Gamma_W(\Psi_{\mathcal{G}_+}^{!-1})) = \mathrm{tr}(R\Gamma_{\tilde{W}}(G_Z), R\Gamma_{\tilde{W}}(\Psi_Z)). \quad (5.57)$$

Proof. By the microlocal Bertini-Sard theorem ([17, Proposition 8.3.12]), there exist $\delta_0, \varepsilon_0 > 0$ such that we have

$$\Lambda \cap \mathbb{R}_{\geq 0}\Lambda_g \cap \pi_M^{-1}(\{0 < |x| \leq \delta_0\}) \subset T_M^*M, \quad (5.58)$$

$$(\Lambda + T_{\{f=0\}}^*M) \cap \mathbb{R}_{\geq 0}\Lambda_g \cap \pi_M^{-1}(\{0 < |x| \leq \delta_0\}) \subset T_M^*M, \quad (5.59)$$

$$\Lambda \cap \mathbb{R}_{\geq 0}\Lambda_f \cap \pi_M^{-1}(\{-\varepsilon_0 \leq f < 0\}) \subset T_M^*M. \quad (5.60)$$

Fix a constant δ such that $0 < \delta < \delta_0$ and set $S := \{x \in M \mid f(x) = 0, |x| = \delta\}$. For $x \in S$ consider covectors $\xi \in \mathbb{R}_{\geq 0}\Lambda_f \cap T_x^*M, \eta \in \mathbb{R}_{\geq 0}\Lambda_g \cap T_x^*M$. Assume that $\xi + \eta = \lambda \in \Lambda$. Then we have $\lambda - \xi = \eta \in (\Lambda + T_{\{f=0\}}^*M) \cap \mathbb{R}_{\geq 0}\Lambda_g$ and thus obtain $\eta = 0$ by (5.59). This implies that we have $\xi = \lambda \in \Lambda \cap \mathbb{R}_{\geq 0}\Lambda_f \cap T_x^*M$ and $\xi = \lambda = 0$. By this argument we obtain

$$\Lambda \cap (\mathbb{R}_{\geq 0}\Lambda_f + \mathbb{R}_{\geq 0}\Lambda_g) \cap T_x^*M \subset \{0\} \quad (5.61)$$

for any $x \in S$. By the compactness of S , there exists an open neighborhood O of S such that for any $x \in O$ we have

$$\mathrm{SS}(G_{\mathcal{G}_+}^{!-1}) \cap (\mathbb{R}_{\geq 0}\Lambda_f + \mathbb{R}_{\geq 0}\Lambda_g) \cap T_x^*M \subset \{0\}, \quad (5.62)$$

$$\mathrm{SS}(R\tau_*G_Z) \cap (\mathbb{R}_{\geq 0}\Lambda_f + \mathbb{R}_{\geq 0}\Lambda_g) \cap T_x^*M \subset \{0\}. \quad (5.63)$$

First, let us prove the equality (5.56). By the microlocal Morse lemma, we have

$$R\Gamma(B(x_0, \delta); G_{\mathcal{G}_+}^{!-1}) \simeq R\Gamma(\overline{B(x_0, \delta)}; G_{\mathcal{G}_+}^{!-1}), \quad (5.64)$$

$$R\Gamma(\tau^{-1}(B(x_0, \delta)); G_Z) \simeq R\Gamma(\tau^{-1}(\overline{B(x_0, \delta)}); G_Z) \quad (5.65)$$

for $0 < \delta < \delta_0$. Thus for $K := \overline{B(x_0, \delta)}$ we obtain

$$\mathrm{tr}(R\Gamma_V(G_{\mathcal{G}_+}^{!-1}), R\Gamma_V(\Psi_{\mathcal{G}_+}^{!-1})) = \mathrm{tr}((G_{\mathcal{G}_+}^{!-1})_K, (\Psi_{\mathcal{G}_+}^{!-1})_K), \quad (5.66)$$

$$\mathrm{tr}(R\Gamma_{\tilde{V}}(G_Z), R\Gamma_{\tilde{V}}(\Psi_Z)) = \mathrm{tr}((G_Z)_{\tilde{K}}, (\Psi_Z)_{\tilde{K}}). \quad (5.67)$$

Moreover by Proposition 5.5 and the local invariance of characteristic classes, we have the equality

$$\mathrm{tr}((G_{\mathcal{G}_+}^{!-1})_K, (\Psi_{\mathcal{G}_+}^{!-1})_K) = \int_{\mathcal{G}} C(G_{\tilde{K}}, \Psi_{\tilde{K}}) \quad (5.68)$$

$$= \int_{\mathcal{G}} C((G_Z)_{\tilde{K}}, (\Psi_Z)_{\tilde{K}}) \quad (5.69)$$

$$= \mathrm{tr}((G_Z)_{\tilde{K}}, (\Psi_Z)_{\tilde{K}}). \quad (5.70)$$

We thus obtain the equality (5.56).

Next let us prove the equality (5.57). For $\varepsilon > 0$ we define an open subset U_ε of W by

$$U_\varepsilon := \{x \in W \mid \mathrm{dist}(x, M \setminus W) > \varepsilon\}. \quad (5.71)$$

For sufficiently small $\varepsilon > 0$, outer conormal vectors of ∂U_ε are contained in $\mathbb{R}_{\geq 0}\Lambda_f + \mathbb{R}_{\geq 0}\Lambda_g \cap \pi_M^{-1}(O)$. Thus by the conditions (5.58), (5.60), (5.62) and (5.63) we can apply the non-characteristic deformation lemma to $G_{g_+}^{!-1}, R\tau_*G_Z$ and the family $\{U_\varepsilon\}_\varepsilon$ to obtain

$$R\Gamma(W; G_{g_+}^{!-1}) \xrightarrow{\sim} R\Gamma(\overline{U}_\varepsilon; G_{g_+}^{!-1}), \quad (5.72)$$

$$R\Gamma(W; R\tau_*G_Z) \xrightarrow{\sim} R\Gamma(\overline{U}_\varepsilon; R\tau_*G_Z) \quad (5.73)$$

for sufficiently small $\varepsilon > 0$. Replacing K by \overline{U}_ε in the proof of (5.56), we obtain the equality (5.57). \square

Applying Lemma 5.7 and Theorem 4.1 to the pair $(R\Gamma_{\tilde{V}}(G_Z), R\Gamma_{\tilde{V}}(\Psi_Z))$, we obtain

$$\mathrm{tr}(R\Gamma_V(G_{g_+}^{!-1}), R\Gamma_V(\Psi_{g_+}^{!-1})) = \sharp([\sigma_f] \cap LC(R\Gamma_{\tilde{V}}(G_Z), R\Gamma_{\tilde{V}}(\Psi_Z))). \quad (5.74)$$

Now define a real analytic function (defined on a neighborhood of $\tau^{-1}(x_0) \subset \mathcal{G}$) $\tilde{g}: \mathcal{G} \rightarrow \mathbb{R}$ by $\tilde{g} := g \circ \tau$. Then by the microlocal Bertini-Sard theorem, there exists $\delta_1 > 0$ such that

$$\mathrm{SS}(G_Z) \cap \Lambda_{\tilde{g}} \cap \pi_{\mathcal{G}}^{-1}(\{v \in \mathcal{G} \mid 0 < |\tau(v)| \leq \delta_1\}) = \emptyset, \quad (5.75)$$

where $\pi_{\mathcal{G}}: T^*\mathcal{G} \rightarrow \mathcal{G}$ is the projection. Moreover by the proof of [17, Theorem 9.5.6], there exists $\delta_2 > 0$ such that

$$c \geq 0, 0 < |x| \leq \delta_2, f(x) > 0 \implies (x; c \cdot dg(x) + df(x)) \notin \Lambda. \quad (5.76)$$

Replacing the constant δ by a smaller one, we may assume that $0 < \delta < \min(\delta_0, \delta_1, \delta_2)$. By the condition (i), (5.75) and the definition of Λ we have

$$\mathrm{supp}(LC(R\Gamma_{\tilde{V}}(G_Z), R\Gamma_{\tilde{V}}(\Psi_Z))) \subset \mathrm{SS}(R\Gamma_{\tilde{V}}(G_Z)) \cap \mathcal{F}_0 \quad (5.77)$$

$$\subset \{\mathrm{SS}(G_Z) \cup (\mathrm{SS}(G_Z) + T_{\partial\tilde{V}}^*\mathcal{G})\} \cap \mathcal{F}_0 \quad (5.78)$$

$$\subset \Lambda \cup (\Lambda + T_{\partial V}^*M) =: \Lambda'. \quad (5.79)$$

Since Λ' is isotropic, by the microlocal Bertini-Sard theorem there exists sufficiently small $\varepsilon_1 > 0$ such that

$$\Lambda' \cap \Lambda_f \cap \pi_M^{-1}(\{0 < |f| \leq \varepsilon_1\}) = \emptyset. \quad (5.80)$$

Arguing as in the proof of [17, Theorem 9.5.6] by using the conditions (5.59), (5.76) and (5.80) and the estimate (5.77)-(5.79), we obtain

$$\Lambda_f \cap \mathrm{supp}(LC(R\Gamma_{\tilde{V}}(G_Z), R\Gamma_{\tilde{V}}(\Psi_Z))) \subset \pi_M^{-1}(\{f < -\varepsilon_1\}) \sqcup \{p_0\}. \quad (5.81)$$

Hence from (5.74) we deduce

$$\begin{aligned} & \mathrm{tr}(R\Gamma_V(G_{g_+}^{!-1}), R\Gamma_V(\Psi_{g_+}^{!-1})) \\ &= \sharp\{\pi_M^{-1}(\{f < -\varepsilon_1\}) \cap [\sigma_f] \cap LC(R\Gamma_{\tilde{V}}(G_Z), R\Gamma_{\tilde{V}}(\Psi_Z))\} + [\sigma_f] \cdot_{p_0} LC(G_Z, \Psi_Z), \end{aligned} \quad (5.82)$$

where $[\sigma_f] \cdot_{p_0} LC(G_Z, \Psi_Z)$ is the local intersection number of $[\sigma_f]$ and $LC(G_Z, \Psi_Z)$ at $p_0 \in \Lambda_i$.

The other term $\mathrm{tr}(R\Gamma_W(G_{g_+}^{!-1}), R\Gamma_W(\Psi_{g_+}^{!-1})) = \mathrm{tr}(R\Gamma_{\widetilde{W}}(G_Z), R\Gamma_{\widetilde{W}}(\Psi_Z))$ can be calculated as follows. For $\varepsilon > 0$, set $W_\varepsilon := W \cap \{f < -\varepsilon\}$ and $\widetilde{W}_\varepsilon := \widetilde{W} \cap \{\tilde{f} < -\varepsilon\} = \tau^{-1}(W_\varepsilon)$.

Lemma 5.8 *There exists sufficiently small $\varepsilon_2 > 0$ such that*

$$\mathrm{tr}(R\Gamma_{\widetilde{W}}(G_Z), R\Gamma_{\widetilde{W}}(\Psi_Z)) = \mathrm{tr}(R\Gamma_{\widetilde{W}_\varepsilon}(G_Z), R\Gamma_{\widetilde{W}_\varepsilon}(\Psi_Z)) \quad (5.83)$$

for any $0 < \varepsilon < \varepsilon_2$.

Proof. Set $\Sigma := \mathrm{SS}(R\Gamma_{\widetilde{V}}(G_Z)) \subset T^*\mathcal{G}$. Then by the microlocal Bertini-Sard theorem there exists $\varepsilon_2 > 0$ such that

$$\Sigma \cap \Lambda_{\widetilde{f}} \cap \pi^{-1}(\{-\varepsilon_2 \leq \widetilde{f} < 0\}) = \emptyset. \quad (5.84)$$

Hence by the microlocal Morse lemma ([17, Corollary 5.4.19]), for $0 < \varepsilon < \varepsilon_2$ we obtain

$$R\Gamma(\{\widetilde{f} < 0\}; R\Gamma_{\widetilde{V}}(G_Z)) \xrightarrow{\sim} R\Gamma(\{\widetilde{f} < -\varepsilon\}; R\Gamma_{\widetilde{V}}(G_Z)). \quad (5.85)$$

□

Let us continue the proof of Proposition 5.6. By Lemma 5.7, Lemma 5.8 and Theorem 4.1, we obtain

$$\mathrm{tr}(R\Gamma_W(G_{\mathcal{G}_+}^{l-1}), R\Gamma_W(\Psi_{\mathcal{G}_+}^{l-1})) = \sharp([\sigma_f] \cap LC(R\Gamma_{\widetilde{W}_\varepsilon}(G_Z), R\Gamma_{\widetilde{W}_\varepsilon}(\Psi_Z))) \quad (5.86)$$

for $0 < \varepsilon < \varepsilon_2$. Moreover it follows from the condition (i) and the definition of Λ that

$$\mathrm{supp}(LC(R\Gamma_{\widetilde{W}_\varepsilon}(G_Z), R\Gamma_{\widetilde{W}_\varepsilon}(\Psi_Z))) \subset \mathrm{SS}(R\Gamma_{\{\widetilde{f} < -\varepsilon\}}(R\Gamma_{\widetilde{V}}(G_Z))) \cap \mathcal{F}_0 \quad (5.87)$$

$$\subset \Lambda' + \mathbb{R}_{\leq 0}\Lambda_f. \quad (5.88)$$

Comparing this last estimate with (5.80), we obtain

$$\Lambda_f \cap \mathrm{supp}(LC(R\Gamma_{\widetilde{W}_\varepsilon}(G_Z), R\Gamma_{\widetilde{W}_\varepsilon}(\Psi_Z))) \subset \pi_M^{-1}(\{f < -\varepsilon_1\}) \quad (5.89)$$

for $0 < \varepsilon < \min(\varepsilon_1, \varepsilon_2)$. Since

$$LC(R\Gamma_{\widetilde{W}_\varepsilon}(G), R\Gamma_{\widetilde{W}_\varepsilon}(\Psi)) = LC(R\Gamma_{\widetilde{V}}(G), R\Gamma_{\widetilde{V}}(\Psi)) \quad (5.90)$$

on $\pi_M^{-1}(\{f < -\varepsilon_1\})$, from (5.86) we obtain

$$\begin{aligned} & \mathrm{tr}(R\Gamma_W(G_{\mathcal{G}_+}^{l-1}), R\Gamma_W(\Psi_{\mathcal{G}_+}^{l-1})) \\ &= \sharp\{\pi_M^{-1}(\{f < -\varepsilon_1\}) \cap [\sigma_f] \cap LC(R\Gamma_{\widetilde{V}}(G_Z), R\Gamma_{\widetilde{V}}(\Psi_Z))\}. \end{aligned} \quad (5.91)$$

Putting (5.82) and (5.91) into (5.55), we finally obtain

$$m_i = [\sigma_f]_{p_0} \cdot LC(G_Z, \Psi_Z), \quad (5.92)$$

which shows

$$LC(G_Z, \Psi_Z) = LC(G_{\mathcal{G}_+}^{l-1}, \Psi_{\mathcal{G}_+}^{l-1}) \quad (5.93)$$

on Λ_i . By the local invariance of Lefschetz cycles, we have

$$LC(G_Z, \Psi_Z) = LC(G, \Psi). \quad (5.94)$$

By combining (5.93) with (5.94) we obtain the assertion. □

Now we return to the situation at the beginning of this section. Namely for a fixed point component M of ϕ assume the conditions $\text{supp}(F) \cap M \subset M_{\text{reg}}$ and

$$“1 \notin \text{Ev}(\phi'_x) \text{ for any } x \in \text{supp}(F) \cap M.” \quad (5.95)$$

Definition 5.9 We define a \mathbb{C} -valued constructible function $\theta(F, \Phi)_M \in \text{CF}(M_{\text{reg}})_{\mathbb{C}}$ on M_{reg} by

$$\theta(F, \Phi)_M = \varphi_{M_{\text{reg}}}(\nu_{M_{\text{reg}}}(F), \Phi'). \quad (5.96)$$

We call it the local trace function of (F, Φ) on the fixed point component M .

By Propositions 5.1 and 5.6 we obtain the following explicit description of the Lefschetz cycle $LC(F, \Phi)_M$.

Theorem 5.10 *We have the equality*

$$LC(F, \Phi)_M = CC(\theta(F, \Phi)_M) \quad (5.97)$$

as Lagrangian cycles in T^*M_{reg} . In particular, if moreover $\text{supp}(F) \cap M$ is compact we have

$$c(F, \Phi)_M = \int_{M_{\text{reg}}} \theta(F, \Phi)_M. \quad (5.98)$$

Corollary 5.11 *Let X , ϕ and M be as above and $F_1 \xrightarrow{\alpha} F_2 \xrightarrow{\beta} F_3 \xrightarrow{\gamma} F_1[1]$ a distinguished triangle in $\mathbf{D}_{\mathbb{R}-c}^b(X)$. Assume that we are given a morphism of distinguished triangles*

$$\begin{array}{ccccccc} \phi^{-1}F_1 & \xrightarrow{\phi^{-1}\alpha} & \phi^{-1}F_2 & \xrightarrow{\phi^{-1}\beta} & \phi^{-1}F_3 & \xrightarrow{\phi^{-1}\gamma} & \phi^{-1}F_1[1] \\ \downarrow \Phi_1 & & \downarrow \Phi_2 & & \downarrow \Phi_3 & & \downarrow \Phi_1[1] \\ F_1 & \xrightarrow{\alpha} & F_2 & \xrightarrow{\beta} & F_3 & \xrightarrow{\gamma} & F_1[1] \end{array} \quad (5.99)$$

in $\mathbf{D}_{\mathbb{R}-c}^b(X)$. Then we have

$$LC(F_2, \Phi_2)_M = LC(F_1, \Phi_1)_M + LC(F_3, \Phi_3)_M. \quad (5.100)$$

In the complex case, we have the following stronger result.

Theorem 5.12 *In the situation of Theorem 5.10, assume moreover that X and $\phi: X \rightarrow X$ are complex analytic and $F \in \mathbf{D}_c^b(X)$ i.e. F is \mathbb{C} -constructible. Then we have*

$$LC(F, \Phi)_M = LC(F|_{M_{\text{reg}}}, \Phi|_{M_{\text{reg}}}) = CC(\varphi(F|_{M_{\text{reg}}}, \Phi|_{M_{\text{reg}}})) \quad (5.101)$$

globally on T^*M_{reg} .

Proof. By Proposition 5.1, we have only to prove

$$LC(\nu_{M_{\text{reg}}}(F), \Phi') = LC(F|_{M_{\text{reg}}}, \Phi|_{M_{\text{reg}}}). \quad (5.102)$$

Since these cycles are considered as sections of the sheaf of $\mathcal{L}_{M_{\text{reg}}}$ of Lagrangian cycles on T^*M_{reg} , it suffices to prove (5.102) locally. Namely, for each $x_0 \in M_{\text{reg}}$ we have only to prove (5.102) on an open neighborhood of $\pi_{M_{\text{reg}}}^{-1}(x_0)$ in $\mathcal{F} \simeq T^*M_{\text{reg}}$. This local statement can be proved along the same line as the proof of Proposition 5.6. Since $\nu_{M_{\text{reg}}}(F)$ admits the action of the multiplicative group $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$ in the complex case, we may use the arguments in the proof of [17, Corollary 9.6.16] for this purpose. This completes the proof. \square

By this theorem we can drop the very technical condition on $\text{supp}(F) \cap M$ in Theorem 2.15.

We give a stalk formula of $\nu_{M_{\text{reg}}}(F)_{\mathcal{E}}^{!-1}$ which is useful to calculate the value of the constructible function $\theta(F, \Phi)_M = \varphi_{M_{\text{reg}}}(\nu_{M_{\text{reg}}}(F), \Phi')$.

Proposition 5.13 *In the situation of Theorem 5.10, let \mathcal{E} be an expanding subbundle of $T_M X$ on a neighborhood of $x \in \text{supp}(F) \cap M \subset M_{\text{reg}}$. Then we have*

$$H^k((\nu_{M_{\text{reg}}}(F)_{\mathcal{E}}^{!-1}))_x \simeq \varinjlim_{B, Z} H_{B \cap Z}^k(B; F) \quad (5.103)$$

for any $k \in \mathbb{Z}$, where B ranges through the family of open neighborhoods of x in X and Z through that of closed subsets of X such that $C_{M_{\text{reg}}}(Z) \cap (\mathcal{E}_x \setminus \{x\}) = \emptyset$.

Proof. By the conicness of $\nu_M(F)$, we have

$$H^k((\nu_M(F)_{\mathcal{E}}^{!-1}))_x \simeq H^k(i_{\mathcal{E}}^!(\nu_M(F)|_{\mathcal{E}}))_x \quad (5.104)$$

$$\simeq H_{\{x\}}^k(\mathcal{E}_x; \nu_M(F)|_{\mathcal{E}_x}). \quad (5.105)$$

In what follows, we use the notation in (2.30). Let B and Z be as in the statement. Since $A := C_M(Z)$ is a closed conic subset of $T_M X$, as in the proof of [17, Theorem 4.2.3 (iii)], we obtain the chain of morphisms:

$$R\Gamma_{B \cap Z}(B; F) \longrightarrow R\Gamma_{p^{-1}(B \cap Z)}(p^{-1}(B); p^{-1}F) \quad (5.106)$$

$$\longrightarrow R\Gamma_{p^{-1}(B \cap Z) \cap \Omega_X}(p^{-1}(B) \cap \Omega_X; p^{-1}F) \quad (5.107)$$

$$\longrightarrow R\Gamma_{(p^{-1}(B \cap Z) \cap \Omega_X) \cup A}(p^{-1}(B) \cap \Omega_X; Rj_* j^{-1} p^{-1}F) \quad (5.108)$$

$$\longrightarrow R\Gamma_{\tau^{-1}(B \cap M) \cap A}(\tau^{-1}(B \cap M); \nu_M(F)). \quad (5.109)$$

Here we used the fact that $(p^{-1}(B \cap Z) \cap \Omega_X) \cup A$ is closed in $p^{-1}(B)$. Since $\mathcal{E}_x \cap A \subset \{x\}$, the restriction morphism induces the one:

$$R\Gamma_{\tau^{-1}(B \cap M) \cap A}(\tau^{-1}(B \cap M); \nu_M(F)) \longrightarrow R\Gamma_{\{x\}}(\mathcal{E}_x; \nu_M(F)|_{\mathcal{E}_x}). \quad (5.110)$$

Combining the above morphisms, we obtain the morphism:

$$\varinjlim_{B, Z} H_{B \cap Z}^k(B; F) \longrightarrow H_{\{x\}}^k(\mathcal{E}_x; \nu_M(F)|_{\mathcal{E}_x}). \quad (5.111)$$

Let us show that this is an isomorphism. The proof is similar to that of [17, Theorem 4.2.3 (iii)]. Set $U = B \setminus Z$ and $V := \tau^{-1}(B \cap M) \setminus C_M(Z)$. Then V is a conic open subset of $T_M X$ and satisfies $\mathring{\mathcal{E}}_x \subset V$, where we set $\mathring{\mathcal{E}}_x := \mathcal{E}_x \setminus \{x\}$. Composing the morphism ([17, Theorem 4.2.3 (ii)])

$$R\Gamma(U; F) \longrightarrow R\Gamma(V; \nu_M(F)), \quad (5.112)$$

and the restriction, we obtain the one

$$R\Gamma(B \setminus Z; F) \longrightarrow R\Gamma(\mathring{\mathcal{E}}_x; \nu_M(F)|_{\mathcal{E}_x}). \quad (5.113)$$

Now consider the following commutative diagram:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \varinjlim_{B, Z} H^{k-1}(B \setminus Z; F) & \longrightarrow & \varinjlim_{B, Z} H^k_{B \cap Z}(B; F) & \longrightarrow & \varinjlim_B H^k(B; F) \longrightarrow \cdots \\ & & \gamma_{k-1} \downarrow & & \alpha_k \downarrow & & \beta_k \downarrow \\ \cdots & \longrightarrow & H^{k-1}(\mathring{\mathcal{E}}_x; \nu_M(F)|_{\mathcal{E}_x}) & \longrightarrow & H^k_{\{x\}}(\mathcal{E}_x; \nu_M(F)|_{\mathcal{E}_x}) & \longrightarrow & H^k(\mathcal{E}_x; \nu_M(F)|_{\mathcal{E}_x}) \longrightarrow \cdots \end{array}$$

Then all the rows are exact and all β_k 's are isomorphisms since $H^k(\mathcal{E}_x; \nu_M(F)|_{\mathcal{E}_x}) \simeq H^k(F)_x$. Thus it suffices to show that γ_k is an isomorphism for any $k \in \mathbb{Z}$. By [17, Theorem 4.2.3 (ii)], we have

$$H^k(\mathring{\mathcal{E}}_x; \nu_M(F)|_{\mathcal{E}_x}) \simeq \varinjlim_W H^k(W; \nu_M(F)) \quad (5.114)$$

$$\simeq \varinjlim_{W, U'} H^k(U'; F), \quad (5.115)$$

where W ranges through conic open neighborhoods of $\mathring{\mathcal{E}}_x$ in $T_M X$ and U' ranges through open subsets of X such that $C_M(X \setminus U') \cap W = \emptyset$. For a pair (B, Z) as in the statement, by taking $U = B \setminus Z$ as U' and $V := \tau^{-1}(B \cap M) \setminus C_M(Z)$ as W , we obtain the morphism

$$\varinjlim_{B, Z} H^k(B \setminus Z; F) \longrightarrow \varinjlim_{W, U'} H^k(U'; F). \quad (5.116)$$

Conversely, for any pair (W, U') as above, if we take any open neighborhood B of x in X and set $Z = X \setminus U'$, we have $C_M(Z) \cap \mathring{\mathcal{E}}_x = \emptyset$ and $B \setminus Z \subset U'$. Hence the morphism (5.116) is an isomorphism. Since the composite of the morphism (5.116) and isomorphisms (5.114)-(5.115) is equal to γ_k , we complete the proof. \square

Remark 5.14 Let $\tau: \mathcal{G} \rightarrow M$ be a vector bundle and $\psi: \mathcal{G} \rightarrow \mathcal{G}$ its endomorphism. Similarly to Definitions 5.2 and 5.3, we can define its minimal shrinking subbundle and shrinking subbundles (cf. [17, Section 9.6]). For a shrinking subbundle \mathcal{S} on an open subset $U \subset M$ and a conic \mathbb{R} -constructible object $G \in \mathbf{D}^b_{\mathbb{R}-c}(\mathcal{G})$, we set

$$G_{\mathcal{S}}^{-1!} := i_{\mathcal{S}}^{-1} j_{\mathcal{S}}^! G \in \mathbf{D}^b_{\mathbb{R}-c}(U), \quad (5.117)$$

where $i_{\mathcal{S}}: U \hookrightarrow \mathcal{S}$ is the zero-section of \mathcal{S} and $j_{\mathcal{S}}: \mathcal{S} \hookrightarrow \mathcal{G}$ is the inclusion map (cf. [1]). Moreover as in Definition 5.4, to a morphism $\Psi: \psi^{-1}G \rightarrow G$ we can associate

its endomorphism $\Psi_S^{-1!}: G_S^{-1!} \longrightarrow G_S^{-1!}$. Then we obtain a result similar to Proposition 5.5 for the pair $(G_S^{-1!}, \Psi_S^{-1!})$ (cf. [17, Proposition 9.6.14]) and can define a constructible function $\varphi_M^s(G, \Psi) \in CF(M)_{\mathbb{C}}$ globally defined on M and associated to it. We can easily show that $\varphi_M^s(G, \Psi) = \varphi_M(G, \Psi)$. Hence we can calculate the value of $\varphi_M(G, \Psi)$ also by shrinking subbundles. In fact the proof of the equality

$$LC(G, \Psi) = CC(\varphi_M^s(G, \Psi)) \quad (5.118)$$

is much easier than that of Proposition 5.6.

6 Some examples

In this section, we introduce various examples to which our results in previous sections are applicable. First of all, we shall give a very simple proof to the following result in [21, Corollary 6.5]. Note that the original proof in [21] relies on some deep results on the functorial properties of Lefschetz cycles (see [21, Sections 5 and 6] for the details). Let M_i be a fixed point component of ϕ such that $\text{supp}(F) \cap M_i \subset (M_i)_{\text{reg}}$. For simplicity, we denote $(M_i)_{\text{reg}}$ by M . Let $M = \sqcup_{\alpha \in A} M_{\alpha}$ be the decomposition of $M = (M_i)_{\text{reg}}$ into its connected components and recall the notations in Introduction.

Theorem 6.1 ([21, Corollary 6.5]) *In addition to the condition:*

$$1 \notin \text{Ev}(\phi'_x) \quad \text{for any } x \in \text{supp}(F) \cap M, \quad (6.1)$$

assume that the inclusion map $i_M: M \hookrightarrow X$ is non-characteristic for F and $\text{supp}(F) \cap M$ is compact. Then we have

$$c(F, \Phi)_M = \sum_{\alpha \in A} \text{sgn}(\text{id} - \phi')_{M_{\alpha}} \cdot \int_{M_{\alpha}} \varphi(F|_{M_{\alpha}}, \Phi|_{M_{\alpha}}). \quad (6.2)$$

Proof. First, we prove the following lemma.

Lemma 6.2 *In the situation of Theorem 6.1, $\nu_M(F)|_{\tau^{-1}(x)}$ is smooth, i.e. its cohomology sheaves are (locally) constant for any $x \in M$.*

Proof. Set $G = \nu_M(F)$. It suffices to show that for any $x \in M$ and $p \in \tau^{-1}(x)$ the restriction morphism

$$R\Gamma(\tau^{-1}(x); G) \longrightarrow G_p \quad (6.3)$$

is an isomorphism. For $p = 0$ it is trivial by the conicness of G . Assume that $p \neq 0$. By the conicness of G we have only to show that for any open convex cone $V \subset T_M X$ containing p and $U = \tau(V)$ the restriction morphism

$$R\Gamma(\tau^{-1}(U); G) \longrightarrow R\Gamma(V; G) \quad (6.4)$$

is an isomorphism. Let $\pi: T_M^* X \longrightarrow M$ be the projection. Then by [17, Proposition 3.7.12] and the isomorphism $G \simeq \mu_M(F)^{\vee}$, the morphism (6.4) is identified with the one

$$R\Gamma_U(\pi^{-1}(U); \mu_M(F)) \longrightarrow R\Gamma_{V^{\circ a}}(\pi^{-1}(U); \mu_M(F)) \quad (6.5)$$

up to some shift. Here we identify U with the zero-section of $\pi^{-1}(U)$ and $a: T_M^*X \rightarrow T_M^*X$ is the antipodal map. From the assumption that $i_M: M \hookrightarrow X$ is non-characteristic for F , we obtain

$$\text{supp}(\mu_M(F)) \subset \text{SS}(F) \cap T_M^*X \subset T_X^*X|_M. \quad (6.6)$$

Thus the morphism (6.5) is an isomorphism. \square

Let us continue the proof of the theorem. We may assume that $M = (M_i)_{\text{reg}}$ is connected. It suffices to show that for any $x \in M$ we have

$$\varphi_M(\nu_M(F), \Phi')(x) = \text{sgn}(\text{id} - \phi')_M \cdot \varphi(F|_M, \Phi|_M)(x). \quad (6.7)$$

We calculate the left hand side of (6.7) by taking a minimal expanding subbundle $\mathcal{W} \subset T_M^*X$ on a sufficiently small neighborhood of x in M . Set $\psi := \phi'|_{\mathcal{W}_x}: \mathcal{W}_x \rightarrow \mathcal{W}_x$ and $H = \nu_M(F)|_{\mathcal{W}_x}$. Then by Lemma 6.2, H is smooth on \mathcal{W}_x and hence we have

$$R\Gamma_c(\mathcal{W}_x; H) \simeq H_x[-d_x] \simeq F_x[-d_x], \quad (6.8)$$

where we set $d_x = \dim \mathcal{W}_x$. Moreover we define a morphism $\Psi: \psi^{-1}H \rightarrow H$ by restricting $\Phi': (\phi')^{-1}\nu_M(F) \rightarrow \nu_M(F)$ to \mathcal{W}_x . Then the left hand side of (6.7) is equal to the trace of the composite of the morphisms

$$R\Gamma_c(\mathcal{W}_x; H) \rightarrow R\Gamma_c(\mathcal{W}_x; \psi^{-1}H) \xrightarrow{\Psi} R\Gamma_c(\mathcal{W}_x; H). \quad (6.9)$$

By the isomorphism (6.8) and $\det \psi > 0$, it is equal to the trace of the composite of the morphisms

$$F_x[-d_x] \simeq (\phi^{-1}F)_x[-d_x] \xrightarrow{\Phi_x} F_x[-d_x]. \quad (6.10)$$

Now the assertion follows immediately from the equality $(-1)^{d_x} = \text{sgn}(\text{id} - \phi')_M$. \square

Remark 6.3 Theorem 6.1 is not true if we do not assume that $i_M: M \hookrightarrow X$ is non-characteristic for F . See e.g. [17, Example 9.6.18].

We have also a lot of examples as follows.

Example 6.4 Let $S^2 = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1\}$ be the 2-dimensional unit sphere in \mathbb{R}^3 and $S^1 = \{e^{i\theta} \mid 0 \leq \theta \leq 2\pi\}$ the 1-dimensional one. Set $X = S^1 \times S^2$. For $e^{i\theta} \in S^1$ we define a real analytic isomorphism $A_\theta: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ of \mathbb{R}^3 by

$$A_\theta(x) = \begin{pmatrix} 2 \cos \theta & -2 \sin \theta & 0 \\ 2 \sin \theta & 2 \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad (6.11)$$

and the one $\phi: X \rightarrow X$ of X by

$$\phi(e^{i\theta}, x) = \left(e^{i\theta}, \frac{A_\theta(x)}{\|A_\theta(x)\|} \right). \quad (6.12)$$

Then the fixed point set M of ϕ is a submanifold of X and consists of 3 connected components M_1, M_2, M_3 defined by

$$M_1 = S^1 \times (0, 0, 1), \quad M_2 = S^1 \times (0, 0, -1), \quad M_3 = \{1\} \times (S^2 \cap \{x_3 = 0\}) \simeq S^1 \quad (6.13)$$

respectively. Note that for $p = (e^{i\theta}, (0, 0, 1)) \in M_1$ the set $\text{Ev}(\phi'_p)$ of the eigenvalues of $\phi'_p: (T_{M_1}X)_p \rightarrow (T_{M_1}X)_p$ is given by $\text{Ev}(\phi'_p) = \{2e^{i\theta}, 2e^{-i\theta}\}$. In particular, it varies depending on the point $p \in M_1$ and satisfies the condition

$$1 \notin \text{Ev}(\phi'_p) \quad \text{for any } p \in M_1. \quad (6.14)$$

Let $\rho: S^2 \setminus \{(0, 0, 1), (0, 0, -1)\} \rightarrow S^1$ be the natural surjective morphism and $I_1, I_2, \dots, I_k \subset S^1$ closed intervals. Assume that I_1, I_2, \dots, I_k are mutually disjoint and

$$\exp\left(\frac{2\pi i}{k}\right) \cdot (I_1 \cup I_2 \cup \dots \cup I_k) = (I_1 \cup I_2 \cup \dots \cup I_k). \quad (6.15)$$

We denote the closure of $\rho^{-1}(I_1 \cup I_2 \cup \dots \cup I_k)$ in S^2 by K . Let us set

$$Y = \left\{ (e^{i\theta}, x) \in X \mid x_3 > \frac{1}{2} \right\}, \quad Z = \{(e^{i\theta}, x) \in Y \mid (e^{i\theta})^k = 1, x \in K\}. \quad (6.16)$$

Then for the constructible sheaf $F = \mathbb{C}_{Y \setminus Z} \in \mathbf{D}_{\mathbb{R}-c}^b(X)$ the inclusion map $i_{M_1}: M_1 \hookrightarrow X$ is characteristic and there exists a natural morphism $\Phi: \phi^{-1}F \rightarrow F$. By Theorem 5.10 we have

$$c(F, \Phi) = c(F, \Phi)_{M_1} = k(k-1). \quad (6.17)$$

Moreover we can easily see that

$$\text{tr}(F, \Phi) = \chi_c(Y) - \chi_c(Z) = 0 - k(1-k) = k(k-1). \quad (6.18)$$

Here $\chi_c(\cdot)$ stands for the Euler characteristic with compact support. Similarly we can construct an example for which the set $\text{Ev}(\phi'_p)$ rotates on a small circle around the point $1 \in \mathbb{C}$. In this case, we cannot take an expanding subbundle of $T_{M_1}X$ globally on M_1 . Such M_1 is not weakly hyperbolic in the sense of Goresky-MacPherson [9] in general.

Example 6.5 Let $\text{Hom}_{\text{gr}}(\mathbb{Z}^n, \mathbb{R} \setminus \{0\})$ be the abelian group consisting of group homomorphisms of the lattice \mathbb{Z}^n to the multiplicative group $\mathbb{R} \setminus \{0\}$. Then the n -dimensional real algebraic torus $T = (\mathbb{R} \setminus \{0\})^n$ can be naturally identified with $\text{Hom}_{\text{gr}}(\mathbb{Z}^n, \mathbb{R} \setminus \{0\})$ (see Fulton [6] etc.). Let $t \in T = \text{Hom}_{\text{gr}}(\mathbb{Z}^n, \mathbb{R} \setminus \{0\})$ be an element of T satisfying the condition:

$$\begin{aligned} &\text{There exists a rational linear subspace } L \subset \mathbb{R}^n \\ &\text{such that } \dim L \geq 1 \text{ and } \text{Ker } t = L \cap \mathbb{Z}^n. \end{aligned} \quad (6.19)$$

Let Σ be a complete smooth fan in \mathbb{R}^n . Assume that there is a cone $\sigma \in \Sigma$ such that

$$\dim \sigma < n \text{ and } \sigma^\perp \subset \mathbb{R} \otimes \text{Ker } t. \quad (6.20)$$

Let $X := X_\Sigma$ be the complete smooth real toric variety associated with Σ and $\phi := l_t: X \rightarrow X$ the natural action of t on it. Then the fixed point set M of ϕ is explicitly given by

$$M = \bigsqcup_{\sigma^\perp \subset \mathbb{R} \otimes \text{Ker } t} T_\sigma, \quad (6.21)$$

where $T_\sigma \simeq (\mathbb{R} \setminus \{0\})^{n-\dim \sigma}$ is the T -orbit associated with the cone $\sigma \in \Sigma$. Define a partial order \preceq on Σ by

$$\sigma \preceq \tau \iff \sigma \text{ is a face of } \tau. \quad (6.22)$$

Let $\sigma_1, \dots, \sigma_k$ be minimal elements of the set $\{\sigma \in \Sigma \mid \sigma^\perp \subset \mathbb{R} \otimes \text{Ker } t\}$ with respect to the order \preceq . Set $M_i := \overline{T_{\sigma_i}} \subset X$. Then M_i is a connected submanifold of X such that $\dim M_i = n - \dim \sigma_i \geq 0$. Moreover $M = \bigsqcup_{i=1}^k M_i$ is the decomposition of M into connected components and we have $\dim M \geq 1$. Let $F \in \mathbf{D}_{\mathbb{R}-c}^b(X)$ be an object satisfying the condition

$$H^j(F|_{T_\sigma}) \text{ is a constant sheaf for any } \sigma \in \Sigma \text{ and } j \in \mathbb{Z}. \quad (6.23)$$

and $\Phi: \phi^{-1}F \rightarrow F$ a morphism in $\mathbf{D}_{\mathbb{R}-c}^b(X)$. Let us apply our fixed point formula to the pair (F, Φ) . For the fixed point component $M_i = \overline{T_{\sigma_i}}$ associated with a minimal cone σ_i we can compute the constructible function $\theta(F, \Phi)_{M_i} = \varphi_{M_i}(\nu_{M_i}(F), \Phi')$ as follows. Set $d := \dim M_i$ and choose an n -dimensional cone $\tau_i \in \Sigma$ such that $\sigma_i \preceq \tau_i$. Then $U_{\tau_i} := \text{Hom}_{\text{gr}}(\tau_i^\vee \cap \mathbb{Z}^n, \mathbb{R}) \simeq \mathbb{R}^n$ is an affine open subset of X containing the T -orbit T_{σ_i} . On $U_{\tau_i} \simeq \mathbb{R}^n$ there exists a coordinate (x_1, x_2, \dots, x_n) such that $M_i \cap U_{\tau_i} = \{x \in \mathbb{R}^n \mid x_{d+1} = \dots = x_n = 0\}$ and the map $\phi = l_t$ can be explicitly written as

$$(x_1, \dots, x_d, x_{d+1}, \dots, x_n) \mapsto (x_1, \dots, x_d, t_{d+1}x_{d+1}, \dots, t_n x_n) \quad (6.24)$$

for some $t_{d+1}, \dots, t_n \neq 0, 1$. Then we can identify $T_{M_i}(X)$ with \mathbb{R}^n on U_{τ_i} . Furthermore, by the condition (6.23), we can identify the pair $(\nu_{M_i}(F), \Phi')$ with (F, Φ) . Interchanging the coordinates, we may assume also that $t_{d+1}, \dots, t_{d+m} > 1, t_{d+m+1}, \dots, t_n < 1$. Then we can take the subbundle

$$\mathcal{E} = \{x \in \mathbb{R}^n \mid x_{d+m+1} = \dots = x_n = 0\} \simeq (M_i \cap U_{\tau_i}) \times \mathbb{R}^m \quad (6.25)$$

as an expanding subbundle of $T_{M_i}(X)$ on $M_i \cap U_{\tau_i}$. Thus, the value of the function $\theta := \theta(F, \Phi)_{M_i}$ at a point $x \in M_i \cap U_{\tau_i}$ is given by

$$\theta(x) = \text{tr}\{R\Gamma_{\{x\}}(\mathcal{E}_x; F|_{\mathcal{E}_x})\} \quad (6.26)$$

$$= \text{tr}(F_x) - \text{tr}\{R\Gamma(\mathcal{E}_x \setminus \{x\}; F)\}, \quad (6.27)$$

where we denote by $\text{tr}\{R\Gamma_{\{x\}}(\mathcal{E}_x; F|_{\mathcal{E}_x})\}$ etc. the trace of the induced endomorphism of $R\Gamma_{\{x\}}(\mathcal{E}_x; F|_{\mathcal{E}_x})$ etc. Note that we have $R\Gamma(\mathcal{E}_x; F|_{\mathcal{E}_x}) \simeq F_x$ by the conicness of $F|_{\mathcal{E}_x}$. We set $S(\mathcal{E}_x) := (\mathcal{E}_x \setminus \{x\})/\mathbb{R}_{>0} \simeq S^{m-1}$ and identify it with the unit sphere of \mathcal{E}_x . Let $\gamma: \mathcal{E}_x \setminus \{x\} \rightarrow S(\mathcal{E}_x)$ the natural map. Then by the conicness of $F|_{\mathcal{E}_x}$, we have

$$\theta(x) = \text{tr}(F_x) - \text{tr}(R\gamma_*(F|_{\mathcal{E}_x \setminus \{x\}}), R\gamma_*(\Phi|_{\mathcal{E}_x \setminus \{x\}})), \quad (6.28)$$

where $R\gamma_*(\Phi|_{\mathcal{E}_x \setminus \{x\}})$ is a natural lift of the map $\gamma_*(\phi|_{\mathcal{E}_x \setminus \{x\}}): S(\mathcal{E}_x) \rightarrow S(\mathcal{E}_x)$. In order to give a more explicit description of the value $\theta(x)$, assume the following condition:

$$t_{d+1}, \dots, t_{d+m} \text{ are distinct.} \quad (6.29)$$

We may assume that $t_{d+1} > \dots > t_{d+m} > 1$. Denote by ξ_1, \dots, ξ_m the corresponding coordinates of $\mathcal{E}_x \simeq \mathbb{R}^m$. Then the fixed point set of $\gamma_*(\phi|_{\mathcal{E}_x \setminus \{x\}})$ is the intersection of the unit sphere with the ξ_i -axes. For $1 \leq i \leq m$ let us calculate the local contribution of the

pair $(R\gamma_*(F|_{\mathcal{E}_x \setminus \{x\}}), R\gamma_*(\Phi|_{\mathcal{E}_x \setminus \{x\}}))$ at $p = (0, \dots, 0, \overset{i}{\pm 1}, 0, \dots, 0) \in S(\mathcal{E}_x)$. The tangent space $V(p) := T_p S(\mathcal{E}_x)$ can be identified with the $(m-1)$ -dimensional linear subspace of \mathbb{R}^m spanned by $\xi_1 \dots \xi_{i-1}, \xi_{i+1}, \dots, \xi_m$ and realized in \mathcal{E}_x as

$$V(p) = \{(\xi_1, \dots, \xi_{i-1}, \pm 1, \xi_{i+1}, \dots, \xi_m) \mid \xi_j \in \mathbb{R}\}. \quad (6.30)$$

Moreover by the condition (6.23), we can identify $\nu_p(R\gamma_*(F|_{\mathcal{E}_x \setminus \{x\}}))$ with $F|_{V(p)}$. Under these identifications, the map on $V(p)$ induced by $\gamma_*(\phi|_{\mathcal{E}_x \setminus \{x\}})$ can be explicitly written as

$$(\xi_1, \dots, \xi_{i-1}, \xi_{i+1}, \dots, \xi_m) \longmapsto (u_1 \xi_1, \dots, u_{i-1} \xi_{i-1}, u_{i+1} \xi_{i+1}, \dots, u_m \xi_m) \quad (6.31)$$

for some $u_1 > \dots > u_{i-1} > 1 > u_{i+1} > \dots > u_m > 0$. Thus at the point p , we can take the $(i-1)$ -dimensional subspace of $T_p S(\mathcal{E}_x)$ spanned by $\xi_1 \dots \xi_{i-1}$ as an expanding subbundle (subspace) and this is realized as

$$W(p) = \{(\xi_1, \dots, \xi_{i-1}, \pm 1, 0, \dots, 0) \mid \xi_j \in \mathbb{R}\}. \quad (6.32)$$

Therefore we have

$$c(R\gamma_*(F|_{\mathcal{E}_x \setminus \{x\}}), R\gamma_*(\Phi|_{\mathcal{E}_x \setminus \{x\}}))_p = \text{tr}\{R\Gamma_{\{p\}}(W(p); F|_{W(p)})\} \quad (6.33)$$

$$= \text{tr}(F_p) - \text{tr}\{R\Gamma(W(p) \setminus \{p\}; F)\}. \quad (6.34)$$

Repeating this argument, we can easily prove the following very simple formula:

$$\theta(x) = \sum_{J \subset \{1, \dots, m\}} \sum_{\substack{\varepsilon = (\varepsilon_j)_{j \in J} \\ \varepsilon_j \in \{\pm 1\}}} (-1)^{|J|} \text{tr}\{R\Gamma(T_{J, \varepsilon}; F|_{T_{J, \varepsilon}})\}, \quad (6.35)$$

where $|J|$ denotes the cardinality of J . Here, for a multi-sign $\varepsilon = (\varepsilon_j)_{j \in J}$, $\varepsilon_j \in \{\pm 1\}$ indexed by J , we set

$$T_{J, \varepsilon} = \{q \in \mathcal{E}_x \simeq \mathbb{R}^m \mid \varepsilon_j q_j > 0 \ (j \in J), \ q_j = 0 \ (j \in \{1, \dots, m\} \setminus J)\}. \quad (6.36)$$

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